Dynamic Programming

M269 Tutorial with Commentary 2023

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4 August 2023

Commentary 1

Agenda, Aims and Topics

1 Agenda, Aims and Topics

- Overview of aims of tutorial
- Note selection of topics
- Recursion is used throughout the topics
- Points about my own background and preferences
- Adobe Connect slides for reference

M269 Dynamic Programming

Tutorial Agenda & Aims

- Welcome and introductions
- Dynamic Programming — introduction
- Python: List comprehensions, Named Tuples
- Implementations in Structured English, Python and Haskell (Optional)
- *Note* there is more material here than we can cover — some is for optional interest
- Not covered in this session: Adobe Connect notes, Fibonacci closed form, Fibonacci alternative calculation Edit Distance — Haskell Implementation, Edit Distance Diagram Construction
- Slides/Notes are at

  pmolyneux.co.uk/OU/M269FolderSync/M269TutorialNotes/M269Tutorial06Prsntn2022JDynamicProgM/
- **Recording** Meeting > Record Meeting... ✔

M269 Tutorial

Introductions — Phil

- *Name* Phil Molyneux
- *Background*
  - Undergraduate: Physics and Maths (Sussex)
  - Postgraduate: Physics (Sussex), Operational Research (Brunel), Computer Science (University College, London)
  - Worked in Operational Research, Business IT, Web technologies, Functional Programming
- *First programming languages* Fortran, BASIC, Pascal
- *Favourite Software*
  - Haskell — pure functional programming language
  - Text editors TextMate, Sublime Text — previously Emacs
  - Word processing in LaTeX — all these slides and notes
  - Mac OS X
- *Learning style* — I read the manual before using the software

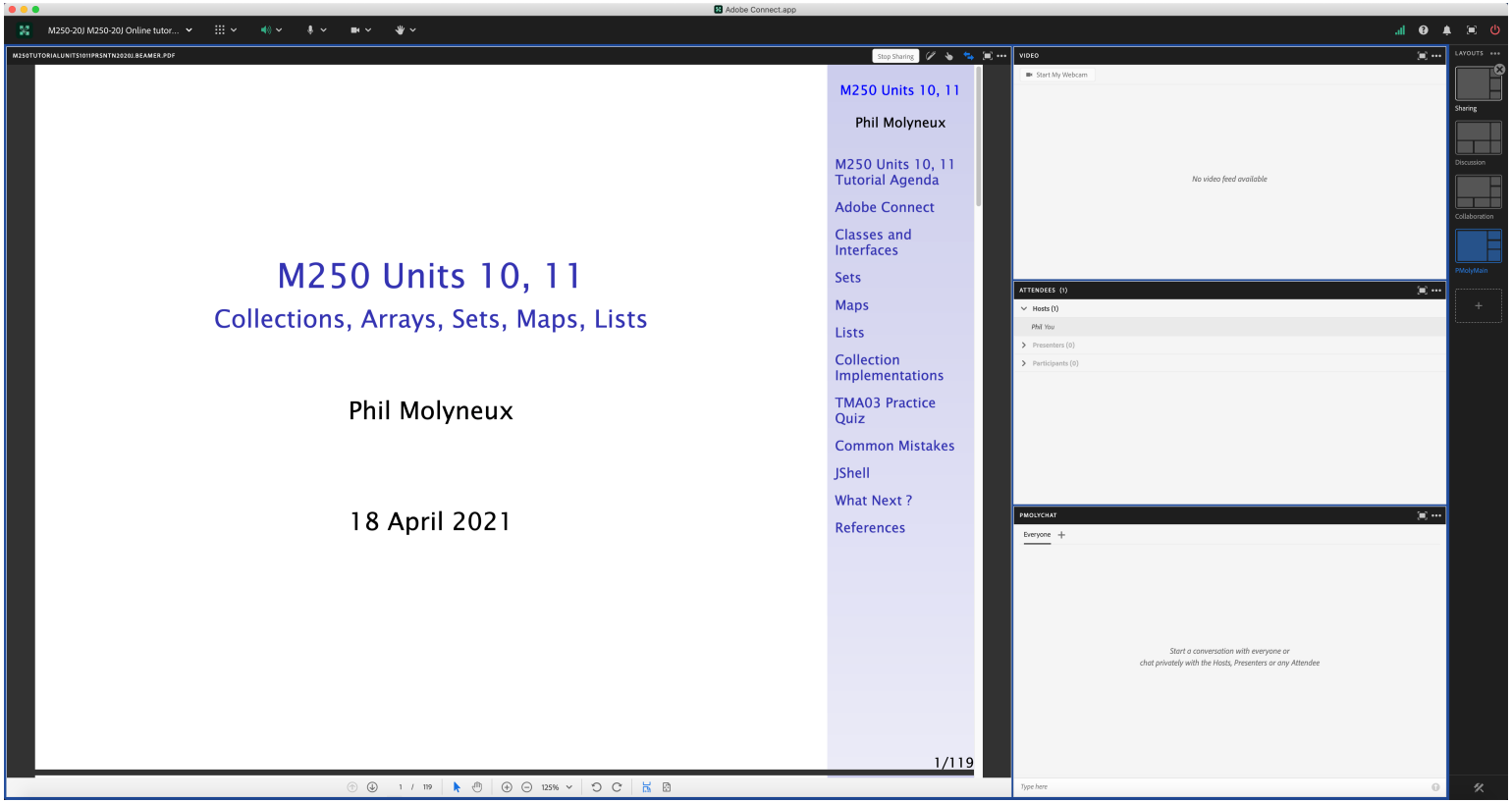
M269 Tutorial

Introductions — You

- *Name ?*
- *Favourite software/Programming language ?*
- *Favourite text editor or integrated development environment (IDE)*
- List of text editors, Comparison of text editors and Comparison of integrated development environments
- *Other OU courses ?*
- *Anything else ?*

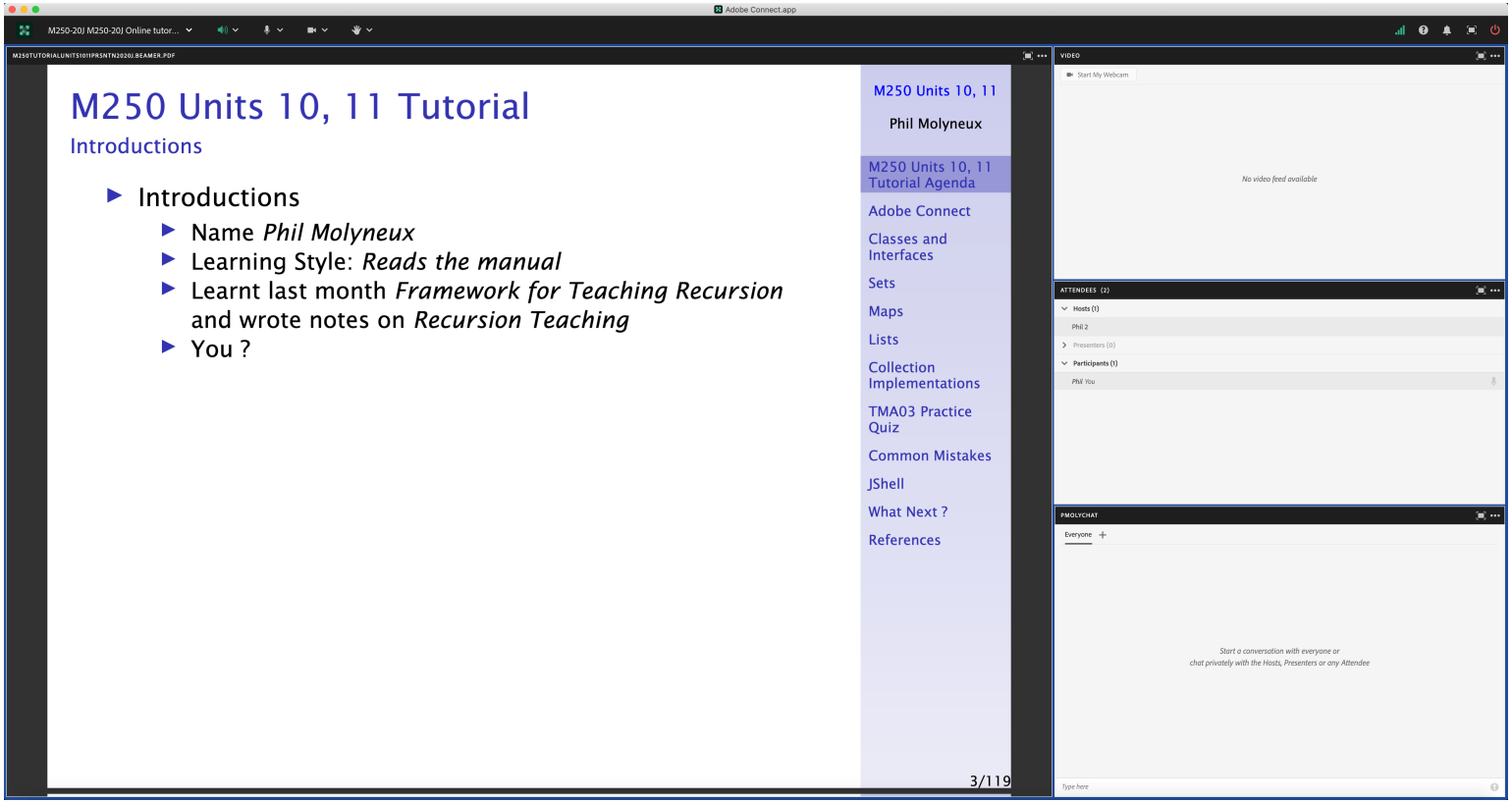
Adobe Connect

Interface — Host View



Adobe Connect

Interface — Participant View



Adobe Connect

Settings

- **Everybody** *Menu bar* > *Meeting* > Speaker & Microphone Setup
- *Menu bar* > *Microphone* > Allow Participants to Use Microphone ✔
- Check Participants see the entire slide **Workaround**
  - *Disable Draw* *Share pod* > *Menu bar* > *Draw icon*
  - *Fit Width* *Share pod* > *Bottom bar* > *Fit Width icon* ✔
- *Meeting* > Preferences > General > **Host Cursor** > Show to all attendees
- *Menu bar* > *Video* > Enable **Webcam** for Participants ✔
- Do not *Enable single speaker mode*
- Cancel hand tool
- Do not enable green pointer
- **Recording** Meeting > Record Session ✔
- **Documents** Upload PDF with drag and drop to share pod
- Delete *Meeting* > Manage Meeting Information > Uploaded Content and *check filename* > *click on delete*

Adobe Connect

Access

- **Tutor Access**

  TutorHome > M269 Website > Tutorials

  Cluster Tutorials > M269 Online tutorial room

  Tutor Groups > M269 Online tutor group room

  Module-wide Tutorials > M269 Online module-wide room

- **Attendance**

  TutorHome > Students > View your tutorial timetables

- **Beamer Slide Scaling** 440% (422 x 563 mm)
- **Clear Everyone's Status**

  Attendee Pod > Menu > Clear Everyone's Status

- **Grant Access** and send link via email

  Meeting > Manage Access & Entry > Invite Participants...

- **Presenter Only Area**

  Meeting > Enable/Disable Presenter Only Area

Adobe Connect

Keystroke Shortcuts

- Keyboard shortcuts in Adobe Connect
- **Toggle Mic** ⌘ + M (Mac), Ctrl + M (Win) (On/Disconnect)
- **Toggle Raise-Hand status** ⌘ + E
- **Close dialog box** ⎋ (Mac), Esc (Win)
- **End meeting** ⌘ + \

Adobe Connect Interface

Sharing Screen & Applications

- Share My Screen > Application tab > Terminal for Terminal
- Share menu > Change View > Zoom in for mismatch of screen size/resolution (Participants)
- (Presenter) Change to 75% and back to 100% (solves participants with smaller screen image overlap)
- Leave the application on the original display
- Beware blued hatched rectangles — from other (hidden) windows or contextual menus
- Presenter screen pointer affects viewer display — beware of moving the pointer away from the application
- First time: System Preferences > Security & Privacy > Privacy > Accessibility

Adobe Connect

Ending a Meeting

- *Notes for the tutor only*
- **Student:** Meeting > Exit Adobe Connect
- **Tutor:**
- **Recording** Meeting > Stop Recording ✔
- **Remove Participants** Meeting > End Meeting... ✔
  - Dialog box allows for message with default message:
  - *The host has ended this meeting. Thank you for attending.*
- **Recording availability** *In course Web site for joining the room, click on the eye icon in the list of recordings under your recording* — edit description and name
- **Meeting Information** Meeting > Manage Meeting Information — can access a range of information in Web page.
- **Delete File Upload** Meeting > Manage Meeting Information > Uploaded Content tab select file(s) and click Delete
- **Attendance Report** see course Web site for joining room

Adobe Connect

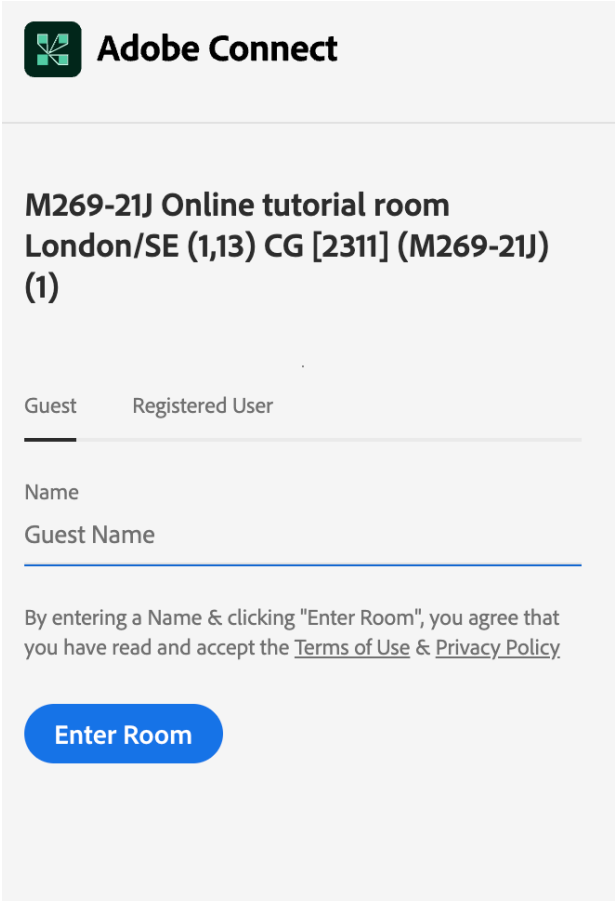
Invite Attendees

- **Provide Meeting URL** Menu > Meeting > Manage Access & Entry > Invite Participants...
- **Allow Access without Dialog** Menu > Meeting > Manage Meeting Information provides new browser window with *Meeting Information* Tab bar > Edit Information
- Check *Anyone who has the URL for the meeting can enter the room*
- Default *Only registered users and accepted guests may enter the room*
- **Reverts to default next session but URL is fixed**
- Guests have blue icon top, registered participants have yellow icon top — same icon if URL is open
- See Start, attend, and manage Adobe Connect meetings and sessions

Adobe Connect

Entering a Room as a Guest (1)

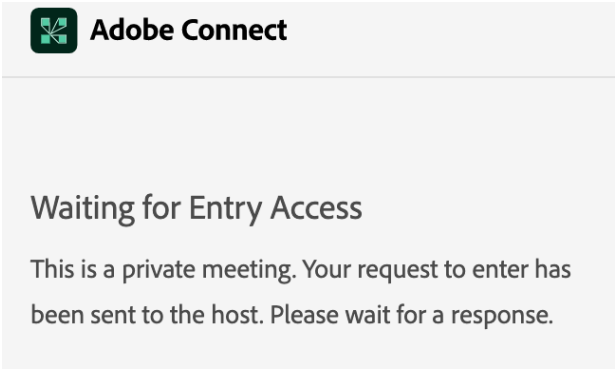
- Click on the link sent in email from the Host
- Get the following on a Web page
- As *Guest* enter your name and click on Enter Room



Adobe Connect

Entering a Room as a Guest (2)

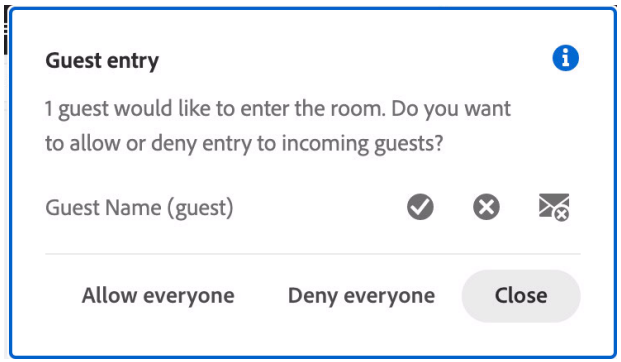
- See the *Waiting for Entry Access* for *Host* to give permission



Adobe Connect

Entering a Room as a Guest (3)

- *Host* sees the following dialog in *Adobe Connect* and grants access



Adobe Connect

Layouts

- **Creating new layouts** example *Sharing* layout
- Menu > Layouts > Create New Layout... > Create a New Layout dialog > Create a new blank layout and name it *PMolyMain*
- New layout has no Pods but does have Layouts Bar open (see Layouts menu)
- **Pods**
- Menu > Pods > Share > Add New Share and resize/position — initial name is *Share n*
- **Rename Pod** Menu > Pods > Manage Pods... > Manage Pods > Select > Rename or Double-click & rename
- Add Video pod and resize/reposition
- Add Attendance pod and resize/reposition
- Add Chat pod — name it *PMolyChat* — and resize/reposition

Adobe Connect

Layouts

- Dimensions of **Sharing** layout (on 27-inch iMac)
  - Width of Video, Attendees, Chat column 14 cm
  - Height of Video pod 9 cm
  - Height of Attendees pod 12 cm
  - Height of Chat pod 8 cm
- **Duplicating Layouts** does *not* give new instances of the Pods and is probably not a good idea (apart from local use to avoid delay in reloading Pods)
- **Auxiliary Layouts** name *PMolyAux0n*
  - Create new Share pod
  - Use existing Chat pod
  - Use same Video and Attendance pods

Adobe Connect

Chat Pods

- **Format Chat text**
- Chat Pod > menu icon > My Chat Color
- Choices: Red, Orange, Green, Brown, Purple, Pink, Blue, Black
- Note: Color reverts to Black if you switch layouts
- Chat Pod > menu icon > Show Timestamps

Graphics Conversion

PDF to PNG/JPG

- Conversion of the screen snaps for the installation of Anaconda on 1 May 2020
- Using GraphicConverter 11
- File > Convert & Modify > Conversion > Convert
- Select files to convert and destination folder
- Click on Start selected Function or ⌘ + ⏎

Adobe Connect Recordings

Exporting Recordings

- *Menu bar* > *Meeting* > Preferences > Video
- Aspect ratio > Standard (4:3) (not Wide screen (16:9) default)
- Video quality > Full HD (1080p not High default 480p)
- **Recording** *Menu bar* > *Meeting* > Record Session ✔
- **Export Recording**
- *Menu bar* > *Meeting* > Manage Meeting Information
- *New window* > Recordings > *check Tutorial* > Access Type button
- check Public > check Allow viewers to download
- **Download Recording**
- *New window* > Recordings > *check Tutorial* > Actions > Download File

Commentary 2

DP Introduction, List Comprehensions

2 DP Introduction, List Comprehensions

- Overview of Dynamic Programming
- Obtain recursive algorithm for problem but implement bottom up
- Introduction to list comprehensions as a concise way of iterating over lists (or other iterables)
- List comprehension exercises with solutions

Dynamic Programming

Introduction (1)

- Dynamic Programming invented by Richard Bellman in the 1950s
- *Programming* meant *planning* the sequence of decisions
- *Dynamic* suggested evolution of the system over time
- Attributes: (1) optimal substructure (2) overlapping subproblems
- Divide and conquer has non-overlapping subproblems — a tree-structure
- DP has an acyclic graph structure

Dynamic Programming

Introduction (2)

- Dynamic Programming process:
- Obtain a recursive solution
- Two ways to avoid subproblems being calculated more than once:
- Memoization uses the top-down recursive structure but preserves subproblems in a table for subsequent retrieval
- Tabulation works bottom-up which orders the computations so that simplest results calculated first and dependent problems follow on in some order

List Comprehensions

Python

- **List Comprehensions** provide a concise way of performing calculations over lists (or other iterables)
- Example: Square the even numbers between 0 and 9

```
Python3>>> [x ** 2 for x in range(10) if x % 2 == 0]
[0, 4, 16, 36, 64]
Python3>>> [(x,y) for x in range(4)
...                for y in range(4)
...                if x % 2 == 0
...                   and y % 3 == 0]
[(0, 0), (0, 3), (2, 0), (2, 3)]
Python3>>>
```

- In general

```
  [expr for target1 in iterable1 if cond1
        for target2 in iterable2 if cond2 ...
        for targetN in iterableN if condN ]
```

- Lots example usage in the algorithms below

List Comprehensions

Haskell

- **List Comprehensions** provide a concise way of performing calculations over lists
- Example: Square the even numbers between 0 and 9

```
GHCi> [x^2 | x <- [0..9], x `mod` 2 == 0]
[0,4,16,36,64]
GHCi>
```

- In general

```
  [expr | qual1, qual2,..., qualN]
```

- The qualifiers `qual` can be
  - Generators `pattern <- list`
  - Boolean guards — acting as filters
  - Local declarations with `let decls` for use in `expr` and later generators and boolean guards

List Comprehension Exercises

Activity 1 (a) Stop Words Filter

- Stop words are the most common words that most search engines avoid: `'a','an','the','that',...`
- Using list comprehensions, write a function `filterStopWords` that takes a list of words and filters out the stop words
- Here is the initial code

```
11  sentence \
12   = "the quick brown fox jumps over the lazy dog"

14  words = sentence.split()

16  wordsTest \
17   = (words == ['the', 'quick', 'brown'
18               , 'fox', 'jumps', 'over'
19               , 'the', 'lazy', 'dog'])

21  stopWords \
22   = ['a','an','the','that']
```

Go to Answer

List Comprehension Exercises

Activity 1 (a) Stop Words Filter

```
11  sentence \
12   = "the␣quick␣brown␣fox␣jumps␣over␣the␣lazy␣dog"

14  words = sentence.split()

16  wordsTest \
17   = (words == ['the', 'quick', 'brown'
18               , 'fox', 'jumps', 'over'
19               , 'the', 'lazy', 'dog'])

21  stopWords \
22   = ['a','an','the','that']
```

- Notice the Python Explicit line joining with (\<nl>) and Python Implicit line joining with (`(...)`)
- The backslash (\) must be followed by an end of line character (<nl>)
- The ('␣') symbol represents a space (see Unicode U+2423 Open Box)

Go to Answer

List Comprehension Exercises

Activity 1 (b) Transpose Matrix

- A matrix can be represented as a list of rows of numbers
- We *transpose* a matrix by swapping columns and rows
- Here is an example

```
38  matrixA \
39   = [[1, 2, 3, 4]
40     ,[5, 6, 7 ,8]
41     ,[9, 10, 11, 12]]

43  matATr \
44   = [[1, 5, 9]
45     ,[2, 6, 10]
46     ,[3, 7, 11]
47     ,[4, 8, 12]]
```

- Using list comprehensions, write a function `transMat`, to transpose a matrix

Go to Answer

List Comprehension Exercises

Activity 1 (c) List Pairs in Fair Order

- Write a function which takes a pair of positive integers and outputs a list of all possible pairs in those ranges
- If we do this in the simplest way we get a bias to one argument
- Here is an example of a bias to the second argument

```
68  yBiasLstTest \
69   = (yBiasListing(5,5)
70       == [(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)
71          , (1, 0), (1, 1), (1, 2), (1, 3), (1, 4)
72          , (2, 0), (2, 1), (2, 2), (2, 3), (2, 4)
73          , (3, 0), (3, 1), (3, 2), (3, 3), (3, 4)
74          , (4, 0), (4, 1), (4, 2), (4, 3), (4, 4)])
```

Go to Answer

List Comprehension Exercises

Activity 1 (c) List Pairs in Fair Order

- Rewrite the function which takes a pair of positive integers and outputs a list of all possible pairs in those ranges
- The output should treat each argument *fairly* — any initial prefix should have roughly the same number of instances of each argument
- Here is an example output

```
81  fairLstTest \
82   = (fairListing(5,5)
83       == [(0, 0)
84           , (0, 1), (1, 0)
85           , (0, 2), (1, 1), (2, 0)
86           , (0, 3), (1, 2), (2, 1), (3, 0)
87           , (0, 4), (1, 3), (2, 2), (3, 1), (4, 0)])
```

Go to Answer

List Comprehension Exercises

Activity 1 (c) List Pairs in Fair Order

- Rewrite the function which takes a pair of positive integers and outputs a list of lists of all possible pairs in those ranges
- The output should treat each argument *fairly* — any initial prefix should have roughly the same number of instances of each argument — further, the output should be segment by each initial prefix (see example below)
- Here is an example output

```
94  fairLstATest \
95   = (fairListingA(5,5)
96       == [[(0, 0)]
97           , [(0, 1), (1, 0)]
98           , [(0, 2), (1, 1), (2, 0)]
99           , [(0, 3), (1, 2), (2, 1), (3, 0)]
100          , [(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)]])
```

Go to Answer

List Comprehension Exercises

Answer 1 (a) Stop Words Filter

- Answer 1 (a) Stop Words Filter
- *Write here:*
- Answer 1 continued on next slide

Go to Activity

List Comprehension Exercises

Answer 1 (a) Stop Words Filter

- Answer 1 (a) Stop Words Filter

```
24  def filterStopWords(words) :
25    nonStopWords \
26     = [word for word in words
27              if word not in stopWords]
28    return nonStopWords


31  filterStopWordsTest \
32   = filterStopWords(words) \
33      == ['quick', 'brown', 'fox'
34        , 'jumps', 'over', 'lazy', 'dog']
```

Go to Activity

List Comprehension Exercises

Answer 1 (b) Transpose Matrix

- Answer 1 (b) Transpose Matrix
- *Write here:*
- Answer 1 continued on next slide

Go to Activity

List Comprehension Exercises

Answer 1 (b) Transpose Matrix

- Answer 1 (b) Transpose Matrix

```
49  def transMat(mat) :
50    rowLen = len(mat[0])
51    matTr \
52     = [[row[i] for row in mat] for i in range(rowLen)]
53    return matTr

55  transMatTestA \
56   = (transMat(matrixA)
57      == matATr)
```

- Note that a list comprehension is a valid expression as a target expression in a list comprehension
- The code assumes every row is of the same length
- Answer 1 continued on next slide

Go to Activity

List Comprehension Exercises

Answer 1 (b) Transpose Matrix

- Note the differences in the list comprehensions below

```
38  matrixA \
39   = [[1, 2, 3, 4]
40     ,[5, 6, 7 ,8]
41     ,[9, 10, 11, 12]]
```

```
Python3>>> [[row[i] for row in matrixA]
...                  for i in range(4)]
[[1, 5, 9], [2, 6, 10], [3, 7, 11], [4, 8, 12]]
Python3>>> [row[i] for row in matrixA
...                 for i in range(4)]
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
Python3>>> [row[i] for i in range(4)
...                 for row in matrixA]
[1, 5, 9, 2, 6, 10, 3, 7, 11, 4, 8, 12]
Python3>>> [[row[i] for i in range(4)]
...                  for row in matrixA]
[[1, 2, 3, 4], [5, 6, 7, 8], [9, 10, 11, 12]]
```

Go to Activity

List Comprehension Exercises

Answer 1 (b) Transpose Matrix

- Answer 1 (b) Transpose Matrix
- The Python NumPy package provides functions for N-dimensional array objects
- For transpose see numpy.ndarray.transpose

```
Python3>>> import numpy as np
Python3>>> ar = np.array([[1,2],[3,4]])
Python3>>> ar
array([[1, 2],
       [3, 4]])
Python3>>> arT = ar.transpose()
Python3>>> arT
array([[1, 3],
       [2, 4]])
Python3>>> ar
array([[1, 2],
       [3, 4]])
Python3>>> ar.shape
(2, 2)
```

Go to Activity

List Comprehension Exercises

Answer 1 (c) List Pairs in Fair Order

- Answer 1 (c) List Pairs in Fair Order — first version
- Write here

```
69  yBiasLstTest \
70   = (yBiasListing(5,5)
71       == [(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)
72           , (1, 0), (1, 1), (1, 2), (1, 3), (1, 4)
73           , (2, 0), (2, 1), (2, 2), (2, 3), (2, 4)
74           , (3, 0), (3, 1), (3, 2), (3, 3), (3, 4)
75           , (4, 0), (4, 1), (4, 2), (4, 3), (4, 4)])
```

Go to Activity

List Comprehension Exercises

Answer 1 (c) List Pairs in Fair Order

- Answer 1 (c) List Pairs in Fair Order
- This is the *obvious* but biased version

```
63  def yBiasListing(xRng,yRng) :
64    yBiasLst \
65     = [(x,y) for x in range(xRng)
66              for y in range(yRng)]
67    return yBiasLst

69  yBiasLstTest \
70   = (yBiasListing(5,5)
71       == [(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)
72           , (1, 0), (1, 1), (1, 2), (1, 3), (1, 4)
73           , (2, 0), (2, 1), (2, 2), (2, 3), (2, 4)
74           , (3, 0), (3, 1), (3, 2), (3, 3), (3, 4)
75           , (4, 0), (4, 1), (4, 2), (4, 3), (4, 4)])
```

Go to Activity

List Comprehension Exercises

Answer 1 (c) List Pairs in Fair Order

- Answer 1 (c) List Pairs in Fair Order — second version
- Write here

```
83  fairLstTest \
84   = (fairListing(5,5)
85       == [(0, 0)
86           , (0, 1), (1, 0)
87           , (0, 2), (1, 1), (2, 0)
88           , (0, 3), (1, 2), (2, 1), (3, 0)
89           , (0, 4), (1, 3), (2, 2), (3, 1), (4, 0)])
```

Go to Activity

List Comprehension Exercises

Answer 1 (c) List Pairs in Fair Order

- Answer 1 (c) List Pairs in Fair Order — second version
- This works by making the sum of the coordinates the same for each prefix

```
77  def fairListing(xRng,yRng) :
78    fairLst \
79     = [(x,d-x) for d in range(yRng)
80                for x in range(d+1)]
81    return fairLst

83  fairLstTest \
84   = (fairListing(5,5)
85       == [(0, 0)
86           , (0, 1), (1, 0)
87           , (0, 2), (1, 1), (2, 0)
88           , (0, 3), (1, 2), (2, 1), (3, 0)
89           , (0, 4), (1, 3), (2, 2), (3, 1), (4, 0)]
```

▸ Go to Activity

List Comprehension Exercises

Answer 1 (c) List Pairs in Fair Order

- Answer 1 (c) List Pairs in Fair Order — third version
- Write here

```
97  fairLstATest \
98   = (fairListingA(5,5)
99       == [[(0, 0)]
100          , [(0, 1), (1, 0)]
101          , [(0, 2), (1, 1), (2, 0)]
102          , [(0, 3), (1, 2), (2, 1), (3, 0)]
103          , [(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)]])
```

Go to Activity

List Comprehension Exercises

Answer 1 (c) List Pairs in Fair Order

- Answer 1 (c) List Pairs in Fair Order — third version
- The *inner loop* is placed into its own list comprehension

```
91  def fairListingA(xRng,yRng) :
92    fairLstA \
93     = [[(x,d-x) for x in range(d+1)]
94                 for d in range(yRng)]
95    return fairLstA

97  fairLstATest \
98   = (fairListingA(5,5)
99       == [[(0, 0)]
100         , [(0, 1), (1, 0)]
101         , [(0, 2), (1, 1), (2, 0)]
102         , [(0, 3), (1, 2), (2, 1), (3, 0)]
103         , [(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)]])
```

Go to Activity

Algorithm Descriptions & Implementations

Python & Haskell Tutorials

- Python tutorials:
  - Beginner's Python Tutorial
  - Python Programming
  - Non-Programmer's Tutorial for Python 3
  - Non-Programmer's Tutorial for Python 2.6
- Haskell Tutorials:
  - Haskell Wikibook
  - What I Wish I Knew When Learning Haskell
  - Haskell Meta-tutorial
  - Learn You a Haskell for Great Good
  - Real World Haskell

Commentary 3

DP, Fibonacci Sequence, Edit Distance example

3 DP Introduction, List Comprehensions

- Fibonacci sequence
- Recursive definition and simple recursive program
- Recursive but more efficient program
- Original implementation in Haskell (optional)
- Implementation in Python
- Edit Distance examples
- Edit Distance diagram construction — generating the LaTeX for the diagrams (optional)

Dynamic Programming

Fibonacci Sequence

- The Fibonacci Numbers or Sequence invented by Leonardo Fibonacci Pisano, who also popularized the Hindu-Arabic numeral system via his 1202 book *Liber Abaci (Book of Calculations)*
- Defined by the recurrence relation
- $F_n = F_{n-1} + F_{n-2}$
- $F_0 = 0, \quad F_1 = 1$ (or originally, $F_1 = 1, \quad F_2 = 1$)
- The Fibonacci numbers have lots of interesting properties
- In programming, often used to illustrate ideas about recurrence relations and recursion

Fibonacci Sequence

Python Recursive (1)

- Recursive definitions are recursive algorithms

```
1  def fibs1(n):
2    return fibs1indent(n,0)
3
4  def fibs1indent(n,indent):
5    indentStr = ' '*indent
6    print(indentStr + 'fibs(' + str(n) + ')')
7
8    if n == 0 :
9      return 0
10   elif n == 1 :
11     return 1
12   else:
13     return (fibs1indent(n - 2, indent + 2)
14           + fibs1indent(n - 1, indent + 2))
```

Fibonacci Sequence

Python Recursive (2)

```
Python3>>> fibs1(5)
fibs(5)
  fibs(3)
    fibs(1)
    fibs(2)
      fibs(0)
      fibs(1)
  fibs(4)
    fibs(2)
      fibs(0)
      fibs(1)
    fibs(3)
      fibs(1)
      fibs(2)
        fibs(0)
        fibs(1)
5
Python3>>>
```

- This take 15 calls to `fibs1()`, 5 calls to `fibs(1)`, 3 calls to `fibs(0)`

Fibonacci Sequence

Python Recursive (2)

- Recursion but remembering enough to avoid most

```
1 def fibs2(n):
2   return fibs2indent(n,0)
3
4 def fibs2indent(n,indent,first=0,second=1):
5   indentStr = ' '*indent
6   print(indentStr + 'fibs(' + str(n) + ')')
7
8   if n == 0 :
9     return [first]
10  else:
11    return ([first]
12            + fibs2indent(n - 1, indent + 2
13                          , second, first + second))
```

Fibonacci Sequence

Python Recursive (2)

```
1 Python3>>> fibs2(5)
2 fibs(5)
3   fibs(4)
4     fibs(3)
5       fibs(2)
6         fibs(1)
7           fibs(0)
8 [0, 1, 1, 2, 3, 5]
9 Python3>>>
```

- This takes 6 calls to `fibs2()`
- `fibs2()` $T(n) = T(n-1) + 1 = n + 1$
- `fibs1()` $T(n) = T(n-1) + T(n-2) + 1 = 2F_{n+1} - 1$
- `fibs1(1)` gets called F_n times
- `fibs1(0)` gets called F_{n-1} times
- So the recursion tree has $F_n + F_{n-1} = F_{n+1}$ leaves
- Since the tree is full it must have $2F_{n+1} - 1$ nodes
- (it is the sum of a geometric series)

Fibonacci Sequence

Closed-form Solution

- $F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$
- Euler-Binet Formula
- A formula for Fib(n)
- ϕ is the Golden mean
- $\phi = \frac{1}{\phi - 1} = \frac{1+\sqrt{5}}{2}$
- Also known as the Golden ratio
- $\phi \stackrel{\text{def}}{=} \frac{a}{b} = \frac{a+b}{a}$

Fibonacci Sequence

Proofs of Euler-Binet Formula (1)

- ProofWiki: Euler-Binet Formula gives 4 proofs:

(1) Proof by induction — straightforward but doesn't really tell you how to get the formula in the first place

(2) Proof by matrix algebra — find the eigenvalues and eigenvectors of a matrix that generates the Fibnacci sequence

(3) Proof from the ProofWiki: Binet Form

$U_n = mU_{n-1} + U_{n-2}$ where $U_0 = 0$ and $U_1 = 1$

(4) Proof from the ProofWiki: Generating Function for Fibonacci Numbers

$$G(z) = \frac{z}{1 - z - z^2}$$

Fibonacci Sequence

Matrix Algebra Proof of Euler-Binet Formula (2)

- Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
- Then $A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$
- Proof by induction
- $A^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix}$
- Assume $A^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$
- Then $A \times A^k = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$

$$= \begin{pmatrix} F_{k+1} + F_k & F_k + F_{k-1} \\ F_{k+1} & F_k \end{pmatrix} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}$$

Fibonacci Sequence

Matrix Algebra Proof of Euler-Binet Formula (3)

- Demonstrate the eigenvalues of A are ϕ and $\hat{\phi} = 1 - \phi$
- Solve $\det(\lambda I - A) = 0$
- $\det\left(\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right) = 0$
- $\det\begin{pmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{pmatrix} = 0$
- $\lambda^2 - \lambda - 1 = 0$
- $ax^2 + bx + c = 0$ has roots $= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Hence $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = 1 - \phi = \frac{1-\sqrt{5}}{2}$
- and $A\begin{pmatrix} \phi \\ 1 \end{pmatrix} = \begin{pmatrix} \phi + 1 \\ \phi \end{pmatrix} = \phi\begin{pmatrix} \phi \\ 1 \end{pmatrix}$ as $\phi^2 - \phi - 1 = 0$
- and $A\begin{pmatrix} \hat{\phi} \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{\phi} + 1 \\ \hat{\phi} \end{pmatrix} = \hat{\phi}\begin{pmatrix} \hat{\phi} \\ 1 \end{pmatrix}$ as $\hat{\phi}^2 - \hat{\phi} - 1 = 0$

Fibonacci Sequence

Matrix Algebra Proof of Euler-Binet Formula (4)

- Hence $\begin{pmatrix} \phi \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue ϕ
- and $\begin{pmatrix} \hat{\phi} \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue $\hat{\phi}$
- We have to have $\det(\lambda I - A) = 0$ nonninvertible, singular
- Proof: assume $X \neq 0$, $AX = \lambda X$ and $(\lambda I - A)$ is invertible
- Then $X = IX = \left((\lambda I - A)^{-1}(\lambda I - A)\right) X$

  $= (\lambda I - A)^{-1}((\lambda I - A)X)$

  $= (\lambda I - A)^{-1} 0$

  $= 0$ contradiction

Fibonacci Sequence

Matrix Algebra Proof of Euler-Binet Formula (5)

- Hence $A^n \begin{pmatrix} \frac{\phi}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \phi^n \begin{pmatrix} \frac{\phi}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$
- and $A^n \begin{pmatrix} \frac{\hat{\phi}}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \hat{\phi}^n \begin{pmatrix} \frac{\hat{\phi}}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$
- Also $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$
- Also $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\phi}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} - \begin{pmatrix} \frac{\hat{\phi}}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$
- Hence $\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \left(\begin{pmatrix} \frac{\phi}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} - \begin{pmatrix} \frac{\hat{\phi}}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \right)$

$$= \phi^n \begin{pmatrix} \frac{\phi}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} - \hat{\phi}^n \begin{pmatrix} \frac{\hat{\phi}}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

Fibonacci Sequence

Matrix Algebra Proof of Euler-Binet Formula (5)

- Hence $A^n \begin{pmatrix} \frac{\phi}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \phi^n \begin{pmatrix} \frac{\phi}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$
- and $A^n \begin{pmatrix} \frac{\hat{\phi}}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \hat{\phi}^n \begin{pmatrix} \frac{\hat{\phi}}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$
- Also $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$
- Also $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\phi}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} - \begin{pmatrix} \frac{\hat{\phi}}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

Fibonacci Sequence

Matrix Algebra Proof of Euler-Binet Formula (6)

- Hence $\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \left(\begin{pmatrix} \frac{\phi}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} - \begin{pmatrix} \frac{\hat{\phi}}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \right)$

$$= \phi^n \begin{pmatrix} \frac{\phi}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} - \hat{\phi}^n \begin{pmatrix} \frac{\hat{\phi}}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} \phi^n \cdot \phi - \hat{\phi}^n \cdot \hat{\phi} \\ \phi^n - \hat{\phi}^n \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} \phi^{n+1} - \hat{\phi}^{n+1} \\ \phi^n - \hat{\phi}^n \end{pmatrix}$$

- Hence $F_n = \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right)$

Fibonacci Sequence

Calculation (1)

- From How to Compute Fibonacci Numbers? and Fibonacci Formulae

```
fib (n + k) = fib (n-1) * fib k + fib n * fib (k+1)
```

- Proof by induction

```
fib (n + 2 + k)
  = fib (n+k) + fib (n+k+1)                          -- by fib
  = fib (n-1) * fib k + fib n * fib (k+1)
    + fib n * fib k + fib (n+1) * fib (k+1)          -- by hyp
  = fib (n+1) * fib k + fib (n+2) * fib (k+1)        -- by fib
```

Fibonacci Sequence

Calculation (2)

- We now derive a method to compute `fib` in O(log n)

```
fib (2n+1) = (fib n)^2 + (fib (n+1))^2    -- (1)
fib (2n+2) = fib n * fib (n+1) + fib (n+1) * fib (n+2)
  = fib n * fib (n+1) + fib (n+1) * (fib n + fib (n+1))
  = 2 * fib n * fib (n+1) + (fib (n+1))^2   -- (2)
fib 2n = 2 * fib n * fib (n+1) - (fib n)^2 -- (3)
  -- (3) = (2) - (1)
```

```
fib2v :: Int -> (Int,Int)
fib2v 0 = (0,1)
fib2v n
  | n `mod` 2 == 0  = (c,d)
  | otherwise       = (d,c+d)
  where
  (a,b) = fib2v (n `div` 2)
  c     = 2 * a * b - a * a
  d     = a * a + b * b
```

Dynamic Programming

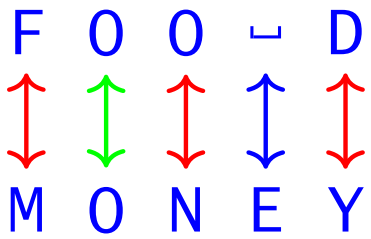
DP Formulation

- Write a recursive algorithm for the problem
- Build solutions to the recurrence from the bottom up
  - Identify subproblems
  - Choose a data structure to memoize intermediate results
  - Identify dependencies between subproblems
  - Find an evaluation order so that each subproblem comes after the subproblems it depends on.
  - Implement the algorithm for the evaluation order

Edit Distance

Definitions

- **Edit Distance** or **Levenshtein Distance** is the minimum number of letter insertions, deletions and substitutions required to transform one word into another.
- Example FOOD → MOOD → MOND → MONED → MONEY
- A better way of displaying the transformation is in the next diagram

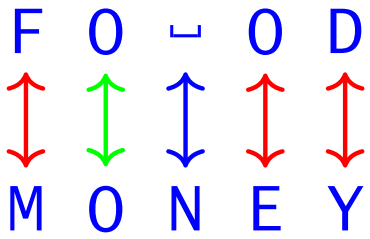


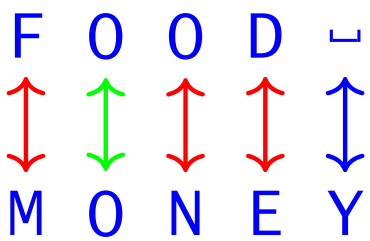
- **Exercise** find two more 4 step transformations. Are there more ?

Edit Distance

Example 1

- Two further transformations of 4 step transformations of FOOD into MONEY are





- How do we find such transformations in general ?

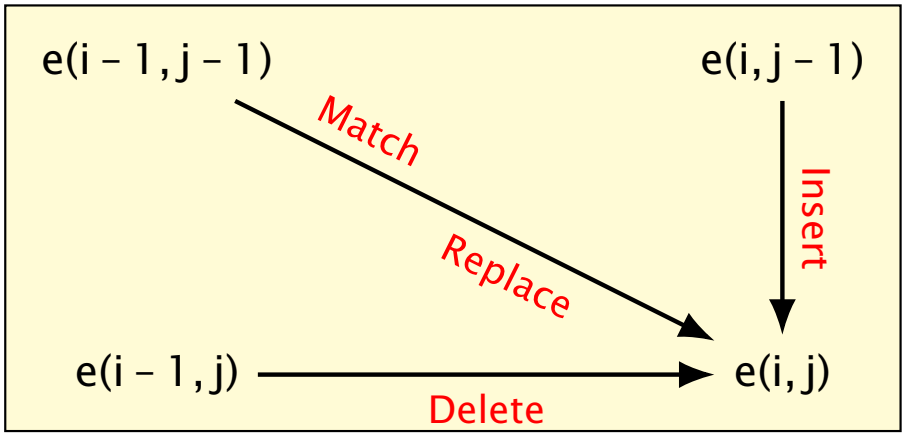
Edit Distance

Recursive Algorithm

- **Notation**
- s and t are names for the *start* and *target* strings
- Let s(i), t(j) denote the *prefixes* of s and t of lengths i and j
- Let s[i], t[j] denote the letters at index i and j of s and t — remember that Python and Haskell index from 0 not 1
- Let e(i, j) denote the edit distance between prefixes s(i) and t(j)
- We now describe recursively how to transform s(i) to t(j)

Edit Distance

Table Dependencies



- **Deletion** — transform s(i – 1) to t(j) and delete the last character of s(i) which is s[i – 1]
- **Insertion** — transform s(i) to t(j – 1) and insert the next character required for t(j) which is t[j – 1]
- **Match/Replace** — transform s(i – 1) to t(j – 1) and match or replace the last character of s(i) with the last character of t(j) that is, s[i – 1] with t[j – 1]
- The edit distance will be the minimum of the three possibilities
- This also determines the predecessor nodes (1 to 3)

Edit Distance

Base Cases & Table Display

- The shortest way to convert s(i) to an empty string ϵ is by i deletions
- The shortest way to convert an empty string ϵ to t(j) is by j insertions
- **Table Display** — note that in the Haskell (but not the Python), the edit distance table is calculated with (i, j) indexing rows and columns but table is displayed with (j, i) indexing rows and columns
- That means that in the table, the row characters are the source and the column characters are the target, while in the display the column characters are the source.
- This seems to be the convention is most texts but may lead to some tricky thinking when formatting the diagrams
- **Note** The next version will change this design decision.

Edit Distance

Recursive Definition

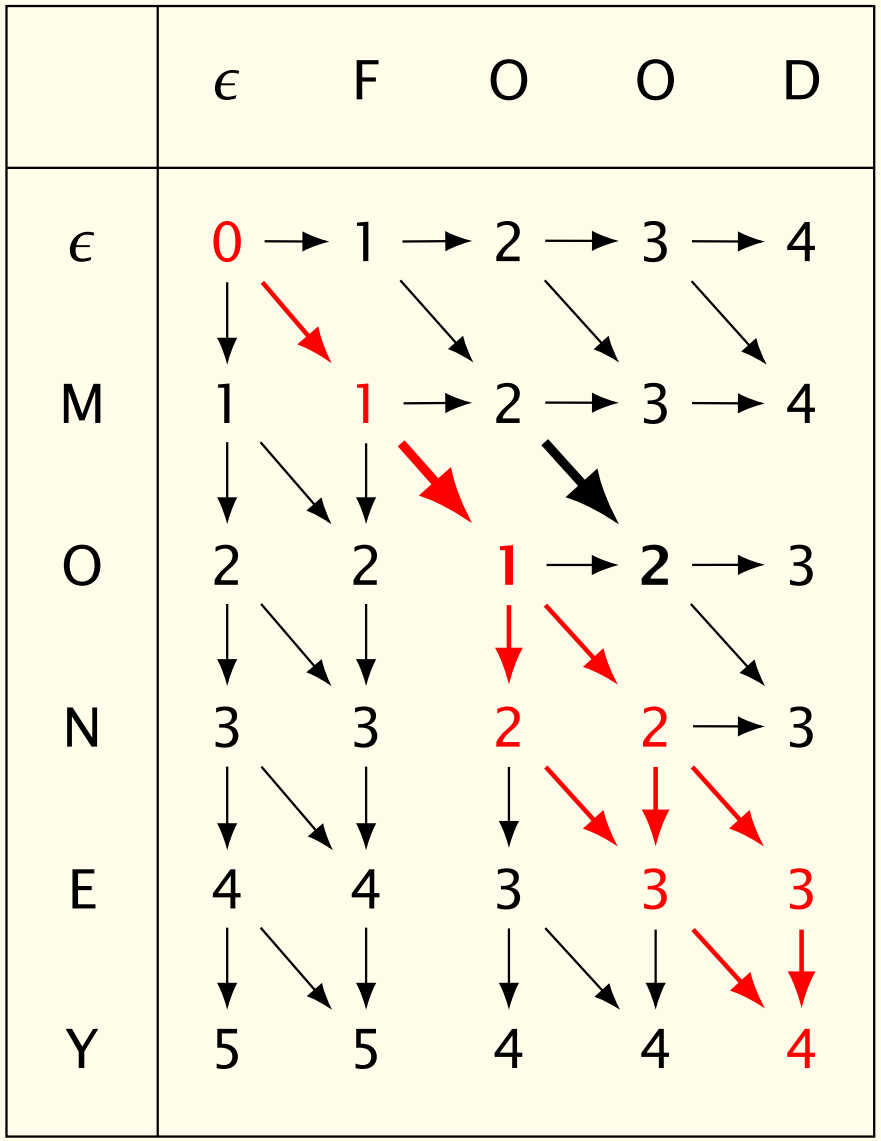
$$e(i,j) = \begin{cases} i & \text{if } j = 0 \\ j & \text{if } i = 0 \\ \min \begin{cases} e(i-1,j)+1 \\ e(i,j-1)+1 \\ e(i-1,j-1)+ \\ \quad \text{if } s[i-1] \neq t[j-1] \text{ then } 1 \text{ else } 0 \end{cases} & \text{otherwise} \end{cases}$$

- For simplicity we have assumed the cost of deletion, insertion, or replacement is 1 and the cost of a match is 0 (made more general in the implementation below)
- the running time of the algorithm is exponential in the length of s and t — but we do not care since are going to implement it bottom up.
- The implementation below also provides the graph of the transformations by calculating the predecessors to each node in the table.

Edit Distance

Memoization Table for `editDistance "FOOD" "MONEY"`

edDistTblFoodMoney



Edit Distance

Dynamic Programming Implementation

- If we calculate the edit distance table in the standard row-major order — row by row, each row from left to right
- then when we reach an entry, the entries it depends on are already available
- We develop the algorithm below in both Haskell and Python
- We have used *list comprehensions* in both for iterations — so they should look fairly similar
- *Note* both implementations could be optimised further
- *Health Warning* you are not expected to write the code for this problem — see the comments on `Python_activity_5.11.py` — this is more of a reading/comprehension exercise

Haskell Implementation

Haskell

```haskell
module M269TutorialDynamicProgCmntry2023 where
  import Data.List
  import Data.Maybe
```

1. A Haskell script starts with a module header which starts with the reserved identifier, `module` followed by the module name, `M269TutorialDynamicProgCmntry2023`
2. The module name must start with an upper case letter and is the same as the file name (without its extension of *.lhs*)
3. Haskell uses *layout* (or the off-side rule) to determine scope of definitions, similar to Python
4. The body of the module follows the reserved identifier `where` and starts with two `import` declarations
5. These import the built-in libraries `Data.List` and `Data.Maybe`

Edit Distance

Haskell Implementation (1)

```
5  type Dist = Int
6  type Pred = (Int,Int)
7  type EditDistCell = (Dist, [Pred])
8  type EditDistTable = [[EditDistCell]]
```

- The reserved identifier `type` introduces *type synonym declarations* to improve readability
- The finite-precision integer type `Int` covers the range $[-2^{63} = -9223372036854775808, 2^{63} - 1 = 9223372036854775807]$ (machine dependent)
- `(t1,t2)` is the type of a pair of `t1` and `t2`
- `[t3]` is the type of a list of elements all of type `t3`
- `[[t3]]` type of a list of lists of elements of type `t3`

Edit Distance

Haskell Implementation — Service Functions (1)

```
10  edCellDist :: EditDistCell -> Dist
11  edCellDist  (x, ps) = x

13  edCellPreds :: EditDistCell -> [Pred]
14  edCellPreds (x, ps) = ps
```

- `edCellDist` and `edCellPreds` take apart an `EditDistCell`

```
f :: t1 -> t2
```

- The above is a type signature for the variable `f`
- This specifies `f` has a function type which takes an element of type `t1` and returns an element of type `t2`
- The definition of `f` may be given without a type signature, in which case the system will infer the most general type
- Note that every function takes exactly one input and returns one output

Edit Distance

Haskell Implementation — Service Functions (2)

```
16  getEdCell :: EditDistTable -> (Int,Int) -> EditDistCell
17  getEdCell edTbl (i,j) = edTbl!!i!!j

19  getEdDist :: EditDistTable -> (Int,Int) -> Dist
20  getEdDist  edTbl (i,j)
21    = edCellDist  (getEdCell edTbl (i,j))

23  getEdPreds :: EditDistTable -> (Int,Int) -> [Pred]
24  getEdPreds edTbl (i,j)
25    = edCellPreds (getEdCell edTbl (i,j))
```

Edit Distance

Haskel points

- Function application is denoted by juxtaposition, is left associative and is more tightly binding than (almost) anything else
  - write `f x` and not `f (x)`
  - say `f x` as *f applied to x*
  - `f x y` means `(f x) y`

  This notational convention has huge advantages — discuss and also see Currying and Functional Programming in 5 Minutes
- To be consistent, the function type arrow `(->)` is right associative
  - `f :: a -> b -> c` means `f :: a -> (b -> c)`
- Lists are denoted `[1,2,3]`, the empty list `[]`
- `(:)` prefixes an element to a list, `1:[2,3] == [1,2,3]`
- Parentheses over-ride precedence
- `(!!)` is the list indexing operator, 0-origin
  - `[1,2,3]!!1 == 2`

Edit Distance

Haskell Implementation — Costs

```
27  -- Cost of deletion, insertion, substitution
28  delC, insC, subC :: Dist
29  delC = 1
30  insC = 1
31  subC = 1
```

- The cost of deletion, insertion and replacement are defined here
- The cost of a match is assumed to be 0

Edit Distance

Haskell Implementation — Algorithm

```
33  editDistTblDP :: String -> String -> EditDistTable
34  editDistTblDP s t
35    = edTbl
36      where
37      edTbl = [ [ edTblCell (i,j) | j <- [0..length t] ]
38                | i <- [0..length s] ]
39      edTblCell (0,0) = (0,[])
40      edTblCell (i,0) = (i * delC, [(i-1,0)])
41      edTblCell (0,j) = (j * insC, [(0,j-1)])
42      edTblCell (i,j)
43        = (minD, preds)
44          where
45          possPreds = [(delD, (i-1,j))
46                      ,(insD, (i,j-1))
47                      ,(subD, (i-1,j-1))
48                      ]
49          delD = getEdDist edTbl (i-1,j) + delC
50          insD = getEdDist edTbl (i,j-1) + insC
51          subD = getEdDist edTbl (i-1,j-1)
52                 + (if s!!(i-1) == t!!(j-1) then 0 else subC)
53          minD = minimum [delD, insD, subD]
54          preds = [(i,j) | (x,(i,j)) <- possPreds, x == minD]
```

Edit Distance

Haskell Implementation — Examples

```
56  edTblTF = editDistTblDP "TREES" "FOREST"
57  edTblTrTF
58    = transpose (editDistTblDP "TREES" "FOREST")

60  edTblAA = editDistTblDP "ALGORITHM" "ALTRUISTIC"
61  edTblTrAA
62    = transpose (editDistTblDP "ALGORITHM" "ALTRUISTIC")

64  edTblFM = editDistTblDP "FOOD" "MONEY"
65  edTblTrFM
66    = transpose (editDistTblDP "FOOD" "MONEY")
```

- We use `transpose` since the edit distance table is displayed with the characters of the start string in columns and the target string in the rows with i indexing the columns and j the rows
- This may lead to some tricky thinking when formatting the diagram but it seems to be the convention.

Edit Distance

Haskell Implementation (2a)

```
67  editDistTblDP01 :: String -> String -> EditDistTable
68  editDistTblDP01 s t
69    = edTbl
70      where
71      edTbl = [ [ setEdTblCell (edTbl,s,t) (i,j)
72                | j <- [0..length t] ]
73                | i <- [0..length s] ]
```

- Defining a subsidiary function `setEdTblCell` avoids having nested `where` clauses
- `setEdTblCell` takes the edit distance table `edTbl`, the start and target strings, and a pair of table indices and returns the table cell.
- The list comprehension definition here is recursive and works because the entries are calculated in row-major order.

Edit Distance

Haskell Implementation (2b)

```
75  setEdTblCell :: (EditDistTable, String, String)
76                  -> (Int, Int) -> EditDistCell
77  setEdTblCell (edTbl,s,t) (0,0) = (0,[])
78  setEdTblCell (edTbl,s,t) (i,0) = (i * delC, [(i-1,0)])
79  setEdTblCell (edTbl,s,t) (0,j) = (j * insC, [(0,j-1)])
80  setEdTblCell (edTbl,s,t) (i,j)
81    = (minD, preds)
82      where
83      possPreds = [(delD, (i-1,j))
84                  ,(insD, (i,j-1))
85                  ,(subD, (i-1,j-1))
86                  ]
87      delD = getEdDist edTbl (i-1,j) + delC
88      insD = getEdDist edTbl (i,j-1) + insC
89      subD = getEdDist edTbl (i-1,j-1)
90             + (if s!!(i-1) == t!!(j-1) then 0 else subC)
91      minD = minimum [delD, insD, subD]
92      preds = [(i,j) | (x,(i,j)) <- possPreds, x == minD]
```

Edit Distance

Haskell Implementation (2c)

```
94  edTblTF01 = editDistTblDP01 "TREES" "FOREST"
95  edTblTrTF01
96    = transpose (editDistTblDP01 "TREES" "FOREST")

98  edTblAA01 = editDistTblDP "ALGORITHM" "ALTRUISTIC"
99  edTblTrAA01
100   = transpose (editDistTblDP01 "ALGORITHM" "ALTRUISTIC")

102 edTblFM01 = editDistTblDP "FOOD" "MONEY"
103 edTblTrFM01
104   = transpose (editDistTblDP01 "FOOD" "MONEY")
```

- `transpose` is used to switch the rows and columns for display
- This may lead to some tricky thinking about formatting the diagram

Edit Distance

Python Implementation (1)

- The Python file `Python_activity_5.11.py` for Python activity 5.11 contains the code for the Dynamic Programming solution for the Levenshtein Edit Distance problem.
- It uses a double `for` loop to do the equivalent calculations of the edit distance table
- TODO rewrite using Python list comprehensions and compare with the Haskell version
- *Note* that the bulk of the code in 5.11 is to format the output and display one path.
- The Haskell to generate the edit table diagrams in these notes is available but not given here since it generates LaTeX TikZ/PGF which is not part of this course.

Edit Distance

Python Implementation (2)

- The Python code below is based on the Haskell code (elsewhere in this document)
- **Health Warning** the code here should be regarded as near pseudo-code since a few of the Haskell features do not translate directly
- In particular, the Haskell code depended to some extent on the default lazy evaluation strategy which is a particular way of implementing non-strict evaluation
- Strict evaluation evaluates arguments of functions (except in special cases or by special constructs)
- Non-strict evaluation evaluates argument to functions at most once and if possible not at all — it evaluates on need
- Python does struct evaluation unless you use yield or other generators
- Haskell is non-strict unless you use strictness annotations
- This means the Python code below may need modifying

Edit Distance

Python Implementation (3)

- Service functions
- Edit Distance Table is an array of cells
- Edit Distance cell is a pair of distance and a list of predecessors

```
21  def edCellDist(cell) :
22    return cell[0]

24  def edCellPreds(cell) :
25    return cell[1]
```

Edit Distance

Python Implementation (4)

- functions to get specific cells
- edTbl is a list of list of cells

```
30  def getEdCell(edTbl, i, j) :
31    return edTbl[i][j]

33  def getEdDist(edTbl, i, j) :
34    return edCellDist(edTbl, i, j)

36  def getEdPreds(edTbl, i, j) :
37    return edCellPreds(edTbl, i, j)
```

Edit Distance

Python Implementation (5)

- Cost of deletion, insertion, substitution

```
41  delC = 1
42  insC = 1
43  subC = 1
```

Edit Distance

Python Implementation (6)

- Calculating the Edit Distance Table
- Calculate the entry in each cell
- setEdTblCell takes the from string, s, the to string, t, the cell indices,i, j, and returns a cell

```
52  def setEdTblCell(edTbl, s, t, i, j) :
53    if i == 0 and j == 0 :
54      return (0,[])
55    elif j == 0 :
56      return (i * delC, [(i-1,0)])
57    elif i == 0 :
58      return (j * insC, [(0,j-1)])
59    else :
60      possPreds = [(delD, (i-1,j))
61                   ,(insD, (i,j-1))
62                   ,(subD, (i-1,j-1))
63                   ]
64      delD = getEdDist(edTbl, i-1,j  ) + delC
65      insD = getEdDist(edTbl, i,  j-1) + insC
66      subD = (getEdDist(edTbl, i-1,j-1)
67              + (0 if s[i-1] == t[j-1] else subC))
68      minD = min([delD, insD, subD])
69      preds = [(i,j) for (x,(i,j)) in possPreds if x == minD]
```

Edit Distance

Python Implementation (5)

- Calculate the Edit Distance Table in row-major order

```
73 def editDistTblDP (s, t) :
74   edTbl = [ [ setEdTblCell (edTbl, s, t, i, j)
75             for j from range(len(t) + 1) ]
76             for i from range(len(s) + 1) ]

78   return edTbl
```

Edit Distance

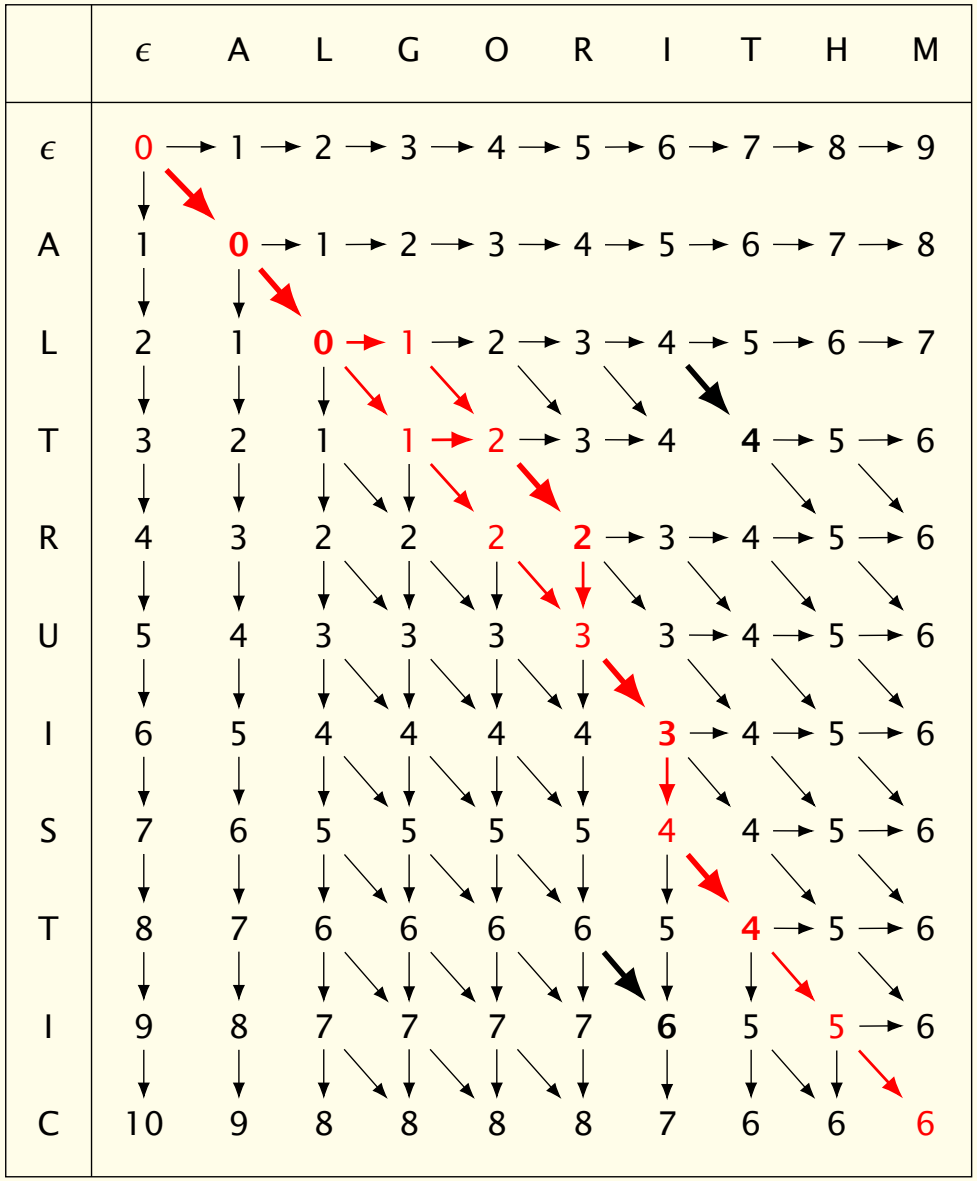
Examples

- For `editDistance "ALGORITHM" "ALTRUISTIC"`
  - What is the edit distance ?
  - How many different routes are there ?
  - Give two examples laid out as in the `editDistance "FOOD" "MONEY"` example
- repeat the above with `editDistance "TREES" "FOREST"` (harder!)

Edit Distance

Memoization Table for `editDistance "ALGORITHM" "ALTRUISTIC"`

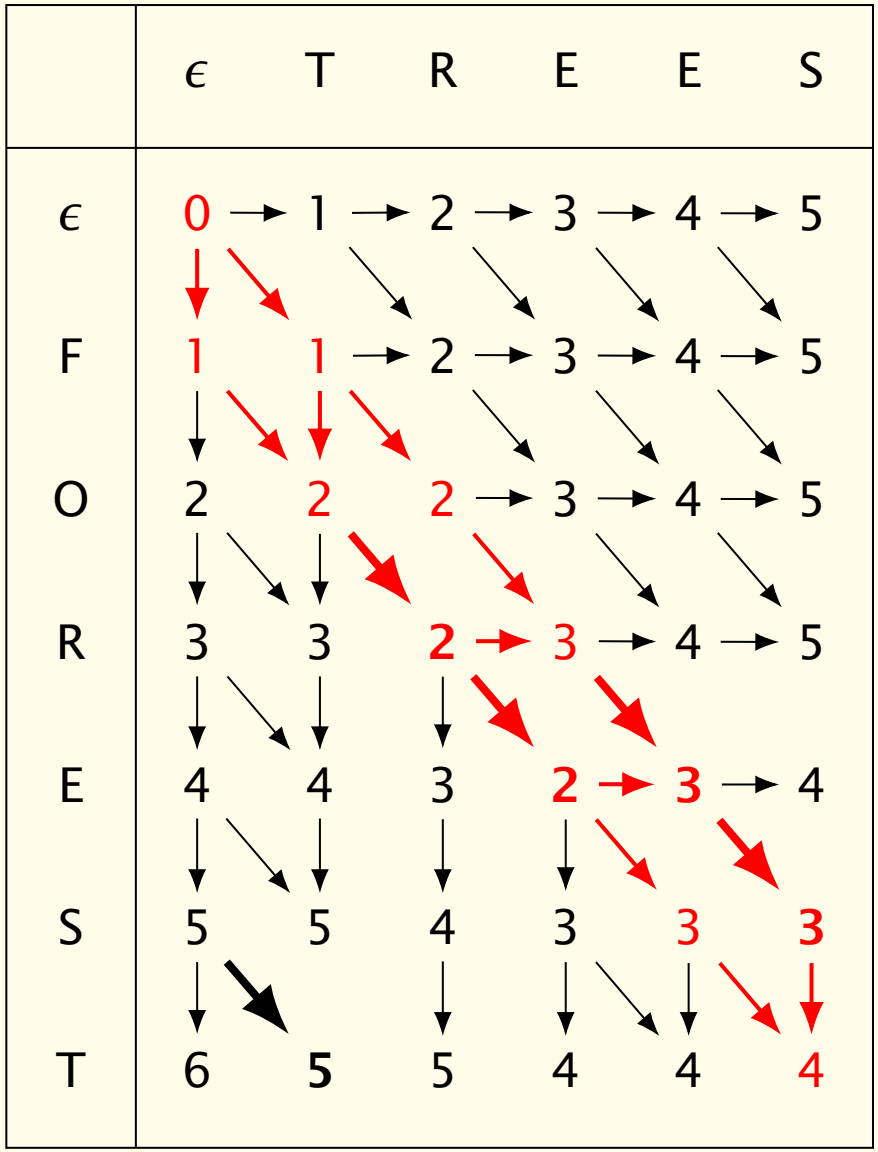
edDistTblAlgorithmAltruistic



Edit Distance

Memoization Table for `editDistance "TREES" "FOREST"`

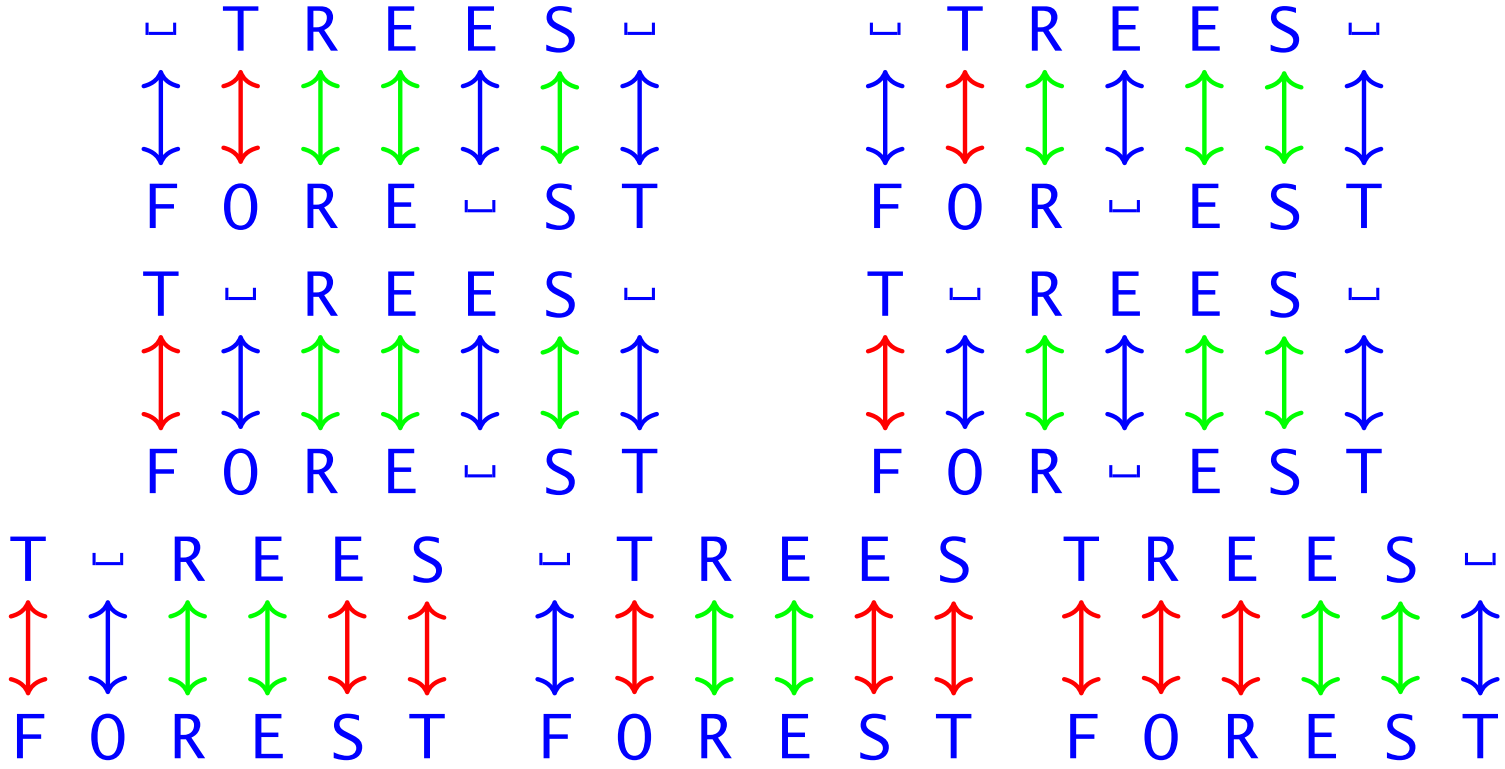
edDistTblTreesForest



Edit Distance

Transformation Operations for `editDistance "TREES" "FOREST"`

- Alternative ways of representing the collection of transformation operations were given on slide 63
- Here is a representation of the edit distance graph with match, delete/insert, and replace color arrows



Edit Distance

Sample Transformations for `editDistance "TREES" "FOREST"`

- The transformations can also be represented as a sequence of operations
- Note that the collection of operations could be used in any order — the end result is the same

Edit Distance

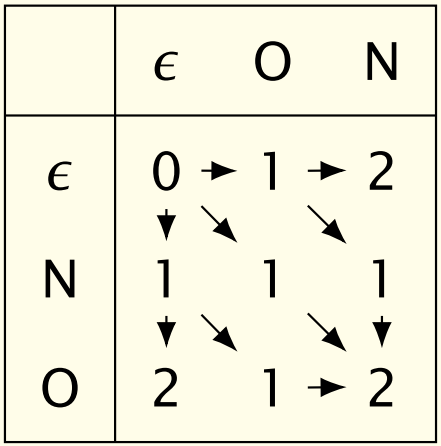
Diagram Construction

- `editDistTblDP` takes two strings and returns an `EditDistTable`
- The diagrams illustrate `EditDistTable` with the cells laid out in a table with arrows
- The diagrams use the markup language LaTeX with the PGF/TikZ package
- Note: the Haskell to LaTeX and TikZ/PGF diagram code is not part of M269 and is only here for interest
- Here are several small examples

Edit Distance

Memoization Table for `editDistTblDP "ON" "NO"`

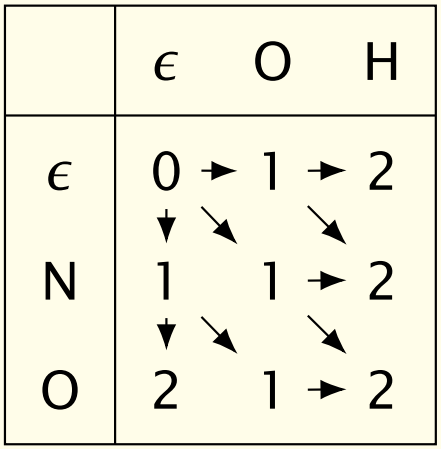
edDistTblOnNo



Edit Distance

Memoization Table for `editDistTblDP "OH" "NO"`

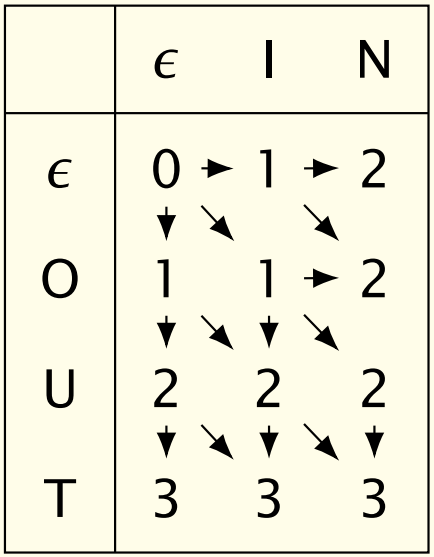
edDistTblOhNo



Edit Distance

Memoization Table for `editDistTblDP "IN" "OUT"`

edDistTblInOut



Edit Distance

LaTeX Code for `editDistTblDP "OH" "NO"`

- The LaTeX code is in a `tikzpicture` environment
- LaTeX TikZ style definitions
- Styles for nodes and arrows
- Edit path nodes and arrows are red and thick
- Free substitutions (matches) have bold nodes and ultra thick arrows

```
2     [ePath/.style={red,thick}
3     ,frSub/.style={font=\bfseries}
4     ,efrSb/.style={ePath,frSub}
5     ,orArr/.style={-Latex}
6     ,frArr/.style={orArr,ultra thick}
7     ,eoArr/.style={ePath,orArr}
8     ,efArr/.style={ePath,frArr}
9     ]
```

Edit Distance

LaTeX Code for `editDistTblDP "OH" "NO"`

- TikZ Matrix code
- The double rows/columns is a hack to ease the frame placement
- See discussion at end

```
% TikZ Matrix code
\matrix (edMat) [matrix of nodes
                ,nodes in empty cells
                ,ampersand replacement=\&
                ,nodes={minimum size=\STtextNodeWidth}]
{
  \&             \&  \&              \&  \&            \&  \&            \& \\
  \&             \&  \&  $\epsilon$ \&  \&          O \&  \&          H \& \\
  \&             \&  \&              \&  \&            \&  \&            \& \\
  \&  $\epsilon$ \&  \&            0 \&  \&          1 \&  \&          2 \& \\
  \&             \&  \&              \&  \&            \&  \&            \& \\
  \&           N \&  \&            1 \&  \&          1 \&  \&          2 \& \\
  \&             \&  \&              \&  \&            \&  \&            \& \\
  \&           O \&  \&            2 \&  \&          1 \&  \&          2 \& \\
  \&             \&  \&              \&  \&            \&  \&            \& \\
};
```

Edit Distance

LaTeX Code for `editDistTblDP "OH" "NO"`

- LaTeX TikZ Arrows code
- All arrows are given the `orArr` style
- TODO: modify the data structure and code to add the edit paths and free substitutions

```
29  % TikZ Arrows code
30  \draw[orArr] (edMat-4-4) -- (edMat-4-6);
31  \draw[orArr] (edMat-4-6) -- (edMat-4-8);

33  \draw[orArr] (edMat-4-4) -- (edMat-6-4);
34  \draw[orArr] (edMat-4-4) -- (edMat-6-6);
35  \draw[orArr] (edMat-6-6) -- (edMat-6-8);
36  \draw[orArr] (edMat-4-6) -- (edMat-6-8);

38  \draw[orArr] (edMat-6-4) -- (edMat-8-4);
39  \draw[orArr] (edMat-6-4) -- (edMat-8-6);
40  \draw[orArr] (edMat-8-6) -- (edMat-8-8);
41  \draw[orArr] (edMat-6-6) -- (edMat-8-8);
```

Edit Distance

LaTeX Code for `editDistTblDP "OH" "NO"`

- LaTeX TikZ Frame code

```
45  % TikZ frame code
46  \draw (edMat-1-1.center) -- (edMat-1-9.center)
47        -- (edMat-9-9.center) -- (edMat-9-1.center)
48        -- cycle;
49  \draw (edMat-3-1.center) -- (edMat-3-9.center);
50  \draw (edMat-1-3.center) -- (edMat-9-3.center);
```

Edit Distance

Generating PGF/TikZ from the Edit Table

- We generate the PGF/TikZ from `editDistTblDP`
- By convention we display the transpose of the table
- Remember that the transpose edit cells have lists of predecessors indexed in the original table
- This makes the computation more awkward but it appears to be the convention
- This computation does not include the edit paths nor the free substitutions
- That would be better included in the edit table computation
- TODO: Include edit paths and free substitutions in the edit table computation

Generating PGF/TikZ from the Edit Table

Offsets, Scales and Constants

```
106  iOffset = 4  -- offset for original string
107  jOffset = 4  -- offset for target string
108  iScale  = 2  -- scale for original string
109  jScale  = 2  -- scale for target string

111  spc = ' '  -- space char
112  nl  = '\n' -- new line char
```

Generating PGF/TikZ from the Edit Table

TikZ Arrow Code (1a)

- The code for drawing arrows between nodes is built on the code for one arrow.
- `edCellTrPredToArrow` takes a pair of style name and matrix name,
- a pair of offset and scale data for original and target string,
- the coordinates of the destination node and a predecessor
- and returns the TikZ code for the arrow

Generating PGF/TikZ from the Edit Table

TikZ Arrow Code (1b)

```
114  edCellTrPredToArrow (styleName,matName)
115    ((iOff,iScl),(jOff,jScl)) (j,i) (a,b)
116   = "\\draw" ++ "[" ++ styleName ++ "]" ++ [spc]
117     ++ "(" ++ matName ++ "-"
118     ++ show (jOff + jScl*b)
119       ++ "-" ++ show (iOff + iScl*a) ++ ")"
120     ++ "␣--␣"
121     ++ "(" ++ matName ++ "-"
122     ++ show (jOff + jScl*j)
123       ++ "-" ++ show (iOff + iScl*i) ++ ");"
124     ++ "\n"
```

Generating PGF/TikZ from the Edit Table

TikZ Arrow Code (2a)

- We use `edCellTrPredToArrow` to build a function to take a transposed edit table and return all the arrows as a TikZ string
- `edCellTrPredToArrow01` is a *helper* function to avoid repeat typing
- `edCellTrPredsToArrows01` takes the list of predecessors in a cell and returns the arrows (as a string)
- `edCellTrToArrows01` takes a cell and returns the arrows
- `edRowTrToArrows01` takes a row and returns the arrows
- `edTblTrToArrows01` takes the complete table and returns the arrows

Generating PGF/TikZ from the Edit Table

TikZ Arrow Code (2b)

```
126  edCellTrPredToArrow01
127   = edCellTrPredToArrow
128       ("orArr","edMat")
129       ((iOffset,iScale),(jOffset,jScale))

131  edCellTrPredsToArrows01  (j,i) ps
132   = concat (map (edCellTrPredToArrow01 (j,i)) ps)

134  edCellTrToArrows01  (j,i) (d,ps)
135   = edCellTrPredsToArrows01 (j,i) ps

137  edRowTrToArrows01 j cs
138   = concat [edCellTrToArrows01 (j,i) c
139            | (i,c) <- zip [0..length cs - 1] cs]
140            ++ "\n"

142  edTblTrToArrows01 edTblTr
143   = concat [edRowTrToArrows01 j cs
144            | (j,cs) <- zip [0..length edTblTr - 1]
145                           edTblTr]
```

Generating PGF/TikZ from the Edit Table

TikZ Arrow Code (2c)

```
147  -- Test Edit Table to Arrows
148  edTblTrToArrows01TF
149   = putStr (edTblTrToArrows01 edTblTrTF)

151  edTblTrToArrows01AA
152   = putStr (edTblTrToArrows01 edTblTrAA)

154  edTblTrToArrows01FM
155   = putStr (edTblTrToArrows01 edTblTrFM)
```

Generating PGF/TikZ from the Edit Table

TikZ Matrix Code (1)

- `edCellTrDistToNode` takes a distance and returns a node string
- `strToNode` takes a string and returns a node string
- **Health Warning** the code below needs rewriting to make it easier to read

Generating PGF/TikZ from the Edit Table

TikZ Matrix Code (1a)

```
157  ndWidth = 13 -- string width of a node
158  -- nWidth = (length zStr + 1) + 2
159  -- spc = ' ' -- spc is defined above

161  ampRepStr = "\\&"

163  zStr = "$\\epsilon$"

165  matrixPreamble nm ampRepStr
166   = "\\matrix␣(" ++ nm ++ ")␣" ++ "[matrix␣of␣nodes"
167     ++ "\n" ++ nSpcs
168     ++ ",nodes␣in␣empty␣cells"
169     ++ "\n" ++ nSpcs
170     ++ ",ampersand␣replacement=" ++ ampRepStr
171     ++ "\n" ++ nSpcs
172     ++ ",nodes={minimum␣size=\\STtextNodeWidth}]"
173     ++ "\n"
174     where
175     nSpcs = replicate (11 + length nm) spc
```

Generating PGF/TikZ from the Edit Table

TikZ Matrix Code (1b)

```
177  -- PGF/TikZ Matrix functions

179  edCellTrDistToNode :: Int -> String
180  edCellTrDistToNode d
181   = replicate n spc ++ dStr ++ [spc]
182     where
183     dStr = show d
184     n    = ndWidth - 1 - length dStr

186  strToNode str
187   = replicate n spc ++ str ++ [spc]
188     where
189     n = ndWidth - 1 - length str
```

Generating PGF/TikZ from the Edit Table

TikZ Matrix Code (2a)

```
191  strToPrefixRow str
192   = "␣␣" ++ ampRepStr
193     ++ replicate ndWidth spc ++ ampRepStr
194     ++ "␣␣" ++ ampRepStr
195     ++ replicate ndWidth spc ++ ampRepStr
196     ++ concat ["␣␣" ++ ampRepStr
197                ++ replicate ndWidth spc ++ ampRepStr
198               | c <- str]
199     ++ "␣\\\\\n"
200     ++ "␣␣" ++ ampRepStr
201     ++ replicate ndWidth spc ++ ampRepStr
202     ++ "␣␣" ++ ampRepStr
203     ++ replicate (ndWidth - 11) spc
204        ++ "$\\epsilon$␣"++ ampRepStr
205     ++ concat ["␣␣" ++ ampRepStr
206                ++ replicate (ndWidth - 2) spc
207                   ++ [c] ++ [spc] ++ ampRepStr
208               | c <- str]
209     ++ "␣\\\\\n"
```

Generating PGF/TikZ from the Edit Table

TikZ Matrix Code (2b)

```
211  strToSuffixRow str
212   = "␣␣" ++ ampRepStr
213     ++ replicate ndWidth spc ++ ampRepStr
214     ++ "␣␣" ++ ampRepStr
215     ++ replicate ndWidth spc ++ ampRepStr
216     ++ concat ["␣␣" ++ ampRepStr
217                ++ replicate ndWidth spc ++ ampRepStr
218               | c <- str]
219     ++ "␣\\\\\n"

221  colNoTrToPrefixCol zStr str j
222   = "␣␣" ++ ampRepStr ++
223       (if j == 0
224       then replicate (ndWidth - 1 - length zStr) spc
225            ++ zStr ++ [spc]
226       else replicate (ndWidth - 2) spc
227           ++ [str!!(j - 1)] ++ [spc])
228     ++ ampRepStr
```

Generating PGF/TikZ from the Edit Table

TikZ Matrix Code (2c)

```
230  edCellTrToNodes (d,ps)
231   = "␣␣" ++ ampRepStr
232     ++ edCellTrDistToNode d ++ ampRepStr

234  edRowTrToNodes zStr str j cs
235   = "␣␣" ++ ampRepStr
236     ++ replicate ndWidth spc ++ ampRepStr
237     ++ concat ["␣␣" ++ ampRepStr
238                ++ replicate ndWidth spc ++ ampRepStr
239               | c <- cs]
240     ++ "␣\\\\\n"
241     ++ colNoTrToPrefixCol zStr str j
242     ++ concat [edCellTrToNodes c | c <- cs]
243     ++ "␣\\\\\n"

245  edTblTrToNodes zStr str edTblTr
246   = concat [edRowTrToNodes zStr str j cs
247             | (j,cs) <- zip [0..length edTblTr - 1]
248                             edTblTr]
```

Generating PGF/TikZ from the Edit Table

TikZ Matrix Code (2cd)

```
250  -- Test usage
251  edTblTrToNodesTF
252   = putStr (edTblTrToNodes zStr "FOREST" edTblTrTF)
```

Generating PGF/TikZ from the Edit Table

TikZ Matrix Code (3a)

```
254  edTblTrToMatrix nm zStr s t
255   = matrixPreamble nm ampRepStr
256     ++ "{\n"
257     ++ strToPrefixRow s
258     ++ edTblTrToNodes zStr t edTblTr
259     ++ strToSuffixRow s
260     ++ "};\n"
261     where
262     edTblTr = transpose (editDistTblDP s t)
```

Generating PGF/TikZ from the Edit Table

TikZ Matrix Code (3b)

```
264  -- Test code for entire matrix
265  edTblTrToMatrixTF
266   = putStr (edTblTrToMatrix "edMat" zStr "TREES" "FOREST")

268  edTblTrToMatrixAA
269   = putStr (edTblTrToMatrix "edMat" zStr "ALGORITHM" "ALTRUISTIC")

271  edTblTrToMatrixFM
272   = putStr (edTblTrToMatrix "edMat" zStr "FOOD" "MONEY")
```

Generating PGF/TikZ from the Edit Table

TikZ Picture Code (1)

```
274  -- Construction of tikzpicture

276  tikzPreamble
277   = "\\begin{tikzpicture}" ++ [nl]
278     ++ "␣␣[ePath/.style={red,thick}" ++ [nl]
279     ++ "␣␣,frSub/.style={font=\\bfseries}" ++ [nl]
280     ++ "␣␣,efrSb/.style={ePath,frSub}" ++ [nl]
281     ++ "␣␣,orArr/.style={-Latex}" ++ [nl]
282     ++ "␣␣,frArr/.style={orArr,ultra␣thick}" ++ [nl]
283     ++ "␣␣,eoArr/.style={ePath,orArr}" ++ [nl]
284     ++ "␣␣,efArr/.style={ePath,frArr}" ++ [nl]
285     ++ "␣␣]"

287  tikzEnd
288   = "\\end{tikzpicture}" ++ [nl]
```

Generating PGF/TikZ from the Edit Table

TikZ Picture Code (2)

```
290  tikzFrmCoord nm loc i j
291   = "(" ++ nm ++ "-" ++ show j ++ "-"
292       ++ show i ++ "." ++ loc ++ ")"

294  tikzFrame nm loc s t
295   = "\\draw " ++ (tikzFrmCoord nm loc 1 1) ++ " -- "
296       ++ (tikzFrmCoord nm loc xMax 1)
297       ++ [nl] ++ dSpcs ++ " -- "
298       ++ (tikzFrmCoord nm loc xMax yMax) ++ " -- "
299       ++ (tikzFrmCoord nm loc 1 yMax)
300       ++ [nl] ++ dSpcs ++ " -- cycle;" ++ [nl]
301     ++ "\\draw " ++ (tikzFrmCoord nm loc 1 (jScale + 1))
302       ++ " -- " ++ (tikzFrmCoord nm loc xMax (jScale + 1))
303       ++ ";" ++ [nl]
304     ++ "\\draw " ++ (tikzFrmCoord nm loc (iScale + 1) 1)
305       ++ " -- " ++ (tikzFrmCoord nm loc (iScale + 1) yMax)
306       ++ ";" ++ [nl]
307     where
308       xMax = iOffset + 1 + iScale * (length s)
309       yMax = jOffset + 1 + jScale * (length t)
310       dSpcs = replicate 6 spc
```

Generating PGF/TikZ from the Edit Table

TikZ Picture Code (3)

```
312  edTblTrToTikz nm loc zStr s t
313   = tikzPreamble ++ [nl]
314     ++ [nl] ++ "%␣TikZ␣Matrix␣code" ++ [nl]
315     ++ edTblTrToMatrix nm zStr s t ++ [nl]
316     ++ [nl] ++ "%␣TikZ␣Arrows␣code" ++ [nl]
317     ++ edTblTrToArrows01 edTblTr ++ [nl]
318     ++ [nl] ++ "%␣TikZ␣frame␣code" ++ [nl]
319     ++ tikzFrame nm loc s t ++ [nl]
320     ++ tikzEnd
321     where
322       edTblTr = transpose (editDistTblDP s t)
```

Generating PGF/TikZ from the Edit Table

Test Code Matrix Frame

```
324  -- Test code for matrix frame lines

326  tikzFrameFM
327   = putStr (tikzFrame "edMat" "center"
328             "FOOD" "MONEY")

330  tikzFrameAA
331   = putStr (tikzFrame "edMat" "center"
332             "ALGORITHM" "ALTRUISTIC")

334  tikzFrameTF
335   = putStr (tikzFrame "edMat" "center"
336             "TREES" "FOREST")
```

Generating PGF/TikZ from the Edit Table

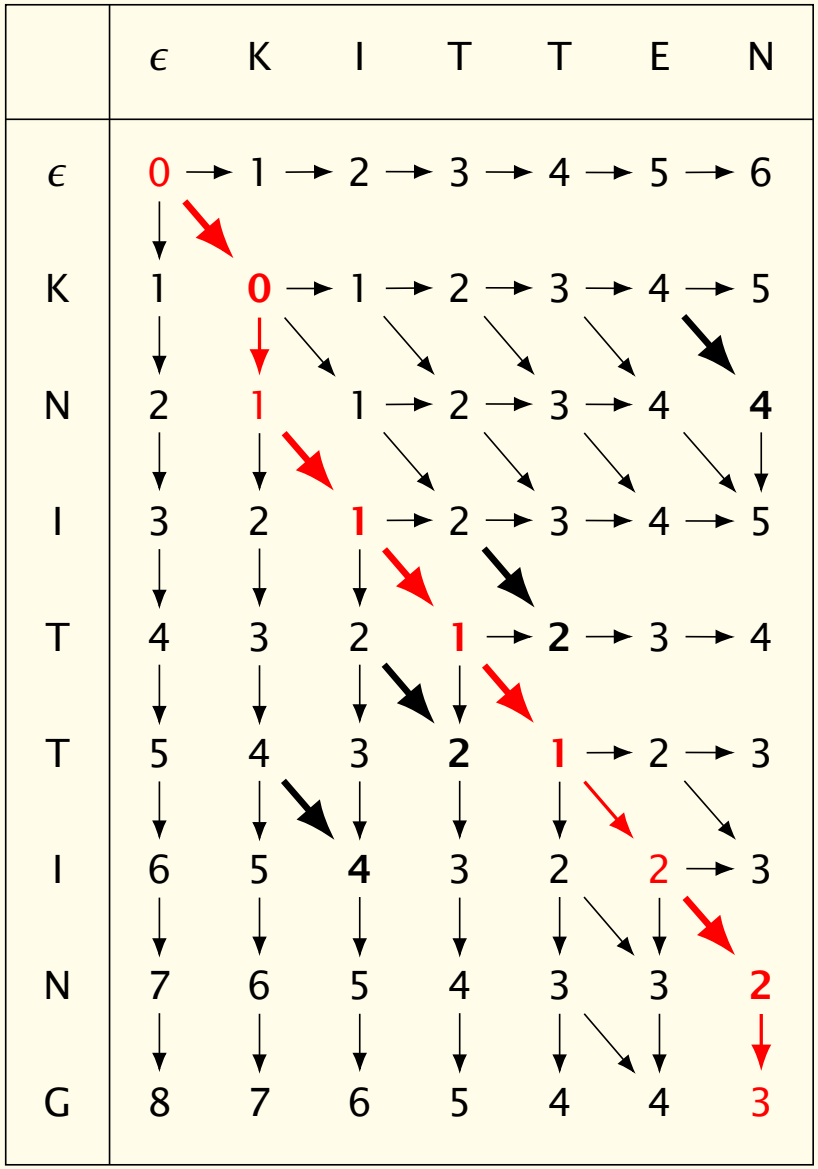
Test Code TikZ Picture

```
338  -- Text code for tikzpicture

340  edTblTrToTikzFM
341   = putStr (edTblTrToTikz "edMat" "center" zStr
342             "FOOD" "MONEY")

344  edTblTrToTikzAA
345   = putStr (edTblTrToTikz "edMat" "center" zStr
346             "ALGORITHM" "ALTRUISTIC")

348  edTblTrToTikzTF
349   = putStr (edTblTrToTikz "edMat" "center" zStr
350             "TREES" "FOREST")

352  edTblTrToTikzKK
353   = putStr (edTblTrToTikz "edMat" "center" zStr
354             "KITTEN" "KNITTING")

356  edTblTrToTikzSE
357   = putStr (edTblTrToTikz "edMat" "center" zStr
358             "SIX" "ELEVEN")
```

Edit Distance

Memoization Table for `editDistance "KITTEN" "KNITTING"`

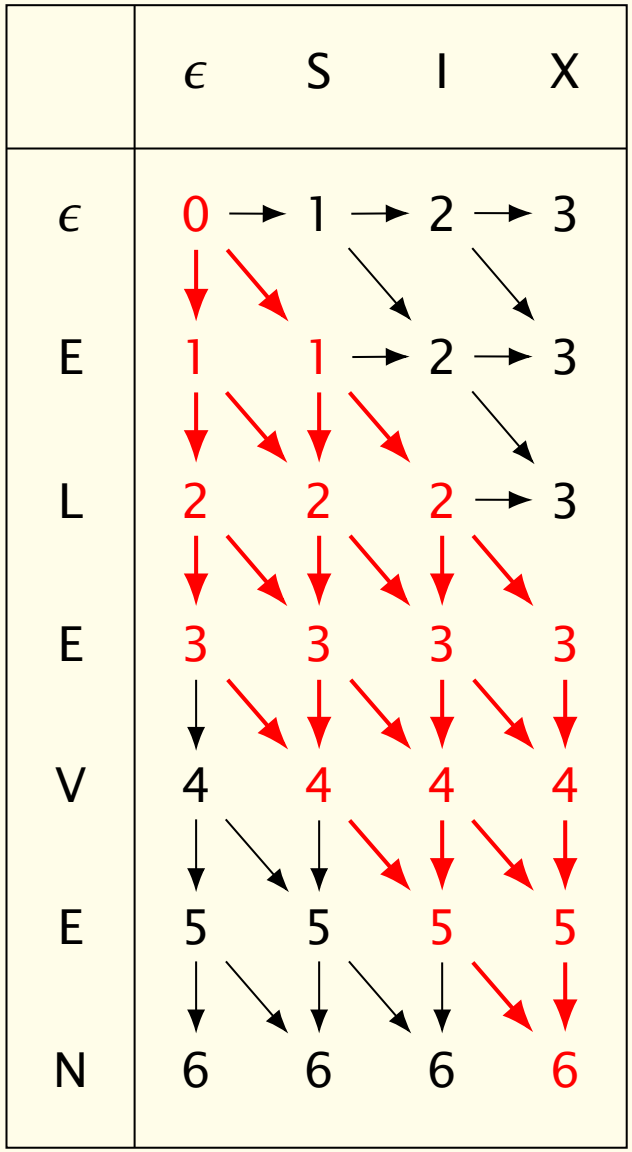
edDistTblKittenKnitting



Edit Distance

Memoization Table for `editDistance "SIX" "ELEVEN"`

edDistTblSixEleven



Commentary 4

Future Work, Other Examples, References

4 Future Work, Other Examples, References

- Future work — Tutorials and TMAs
- Other examples
- References and other sources
- **Colophon**
- LaTeX with Beamer, Listings and other packages
- Index of Python code and diagrams
- PGF/TikZ for the diagrams
- External copies of the diagrams as PDF with tight bounding boxes are available

Future Work

Topics & Events

- Future tutorial and TMA dates
- **Other DP examples**
- Longest common subsequence
- Knapsack
- TSP

Edit Distance

Sources

- Jeff Erickson Algorithms — basis of these notes
- Wikibooks Levenshtein
- Wikipedia: Levenshtein
- Linux Magazine March 2016 Issue 184 Better Finds — article on egrep which uses the Levenshtein algorithm
- Rosetta Code: Levenshtein Distance
- Peter Norvig How to Write a Spelling Corrector
- Stack Overflow: Edit Distance in Haskell
- *Levenshtein Algorithm* www.levenshtein.net
- *Levenshtein Demo* let.rug.nl/~kleiweg/lev/
- Edit distance (Levenshtein-Distance) algorithm explanation
- Wikipedia: Damerau-Levenshtein distance allows for transposition of two adjacent characters

Algorithms

Texts & Web Sites

- Jeff Erickson Algorithms
- Cormen et al (2022) *Introduction to Algorithms*
- Sedgewick (2011) *Algorithms* see Algorithms, 4th edition
- Bird, Gibbons (2020) *Algorithm Design with Haskell*

Analysis of Algorithms

Big-O

- Big O notation
- The Algebra of Big-O
- Python Time Complexity — also summarised in Software Summaries
- Computational Complexity of a List Comprehension — as it says: *The structure of a list comprehension makes it easy to determine computational complexity.* This is from An Introduction to Functional Programming, Lazy Evaluation, and Streams in Python (10 August 2023)
- Big-O Cheat Sheet

Graph Algorithms

References

- Graph Theory: Trees
- Graph Theory
- Dekai Wu Algorithms course
- http://jeffe.cs.illinois.edu/teaching/algorithms/Jeff Erickson Algorithms

Shortest Paths

References

- Dijkstra's Algorithm
  - Dijkstra's shortest path algorithm
- Bellman-Ford
  - Bellman–Ford Algorithm
  - Bellman Ford Algorithm (Simple Implementation)
  - Haskell Hackage Data.BellmanFord
  - Data.IGraph
    from igraph
  - igraph Reference Manual Chapter 13 Structural Properties of Graphs
  - R igraph manual pages: Shortest (directed or undirected) paths between vertices
  - IGraph/M: a Mathematica interface for igraph