Binary Trees

M269 Module-wide Tutorial

Contents

Commentary 1

1 Agenda, Aims and Topics

- Overview of aims of tutorial
- Note selection of topics
- Recursion is used throughout the topics
- Points about my own background and preferences
- Adobe Connect slides for reference

ToC

1 M269 Tutorial Agenda — Binary Trees, Recursion, Searching

1. Welcome and introductions
2. To cover some of
    - Binary Trees
    - Binary Search Trees
    - Height Balanced (AVL) Trees

3. Questions & discussion (at any point)
4. *Adobe Connect* — if you or I get cut off, wait till we reconnect (or send you an email)
5. *Source:* of slides, notes, programs:

   M269Tutorial20260208BinaryTreesPrsntn2025JM/

6. **Python Files** M269Tutorial20260208BinaryTreesPrsntn2025JM/Python/

- There is a lot more material in these slides/notes than we can cover in the available time, so I will cover:

(1) Binary Tree terminology and representation — some choices

(2) Tree traversal — depth first recursive

(3) Tree traversal — breadth first — recursive first, transformed to the usual iterative version

- These notes are as much about recursion as Binary trees — the notes give several examples of evaluations and what to do when you make a mistake

(4) Binary search trees — deleting a node — choices

(5) AVL or height balanced trees — brief introduction

Health Warning These notes contain some material that is not part of M269 but is present for interest

Tutorial Materials

- From the Web link to the folder containing the tutorial materials you should find:
- File with name ending `.beamer.pdf` — the slides
- File with name ending `.article.pdf` — the notes version
- Table of contents — in the slides this is a clickable sidebar; in the notes it is an expanded list of sections with links from the end of sections
- Indices — the notes version has an index of the Python code and the diagrams
- References — the notes version has references which have back references to the pages where the reference is cited

Introductions — Phil

- *Name* Phil Molyneux
- *Background*
  - Undergraduate: Physics and Maths (Sussex)
  - Postgraduate: Physics (Sussex), Operational Research (Brunel), Computer Science (University College, London)
  - Worked in Operational Research, Business IT, Web technologies, Functional Programming
- *First programming languages* Fortran, BASIC, Pascal

- *Favourite Software*
  - Haskell — pure functional programming language
  - Text editors TextMate, Sublime Text — previously Emacs
  - Word processing in $\LaTeX$ — all these slides and notes
  - Mac OS X
- *Learning style* — I read the manual before using the software

Introductions — You

- *Name* ?
- *Favourite software/Programming language* ?
- *Favourite text editor or integrated development environment (IDE)*
- List of text editors, Comparison of text editors and Comparison of integrated development environments
- *Other OU courses* ?
- *Anything else* ?

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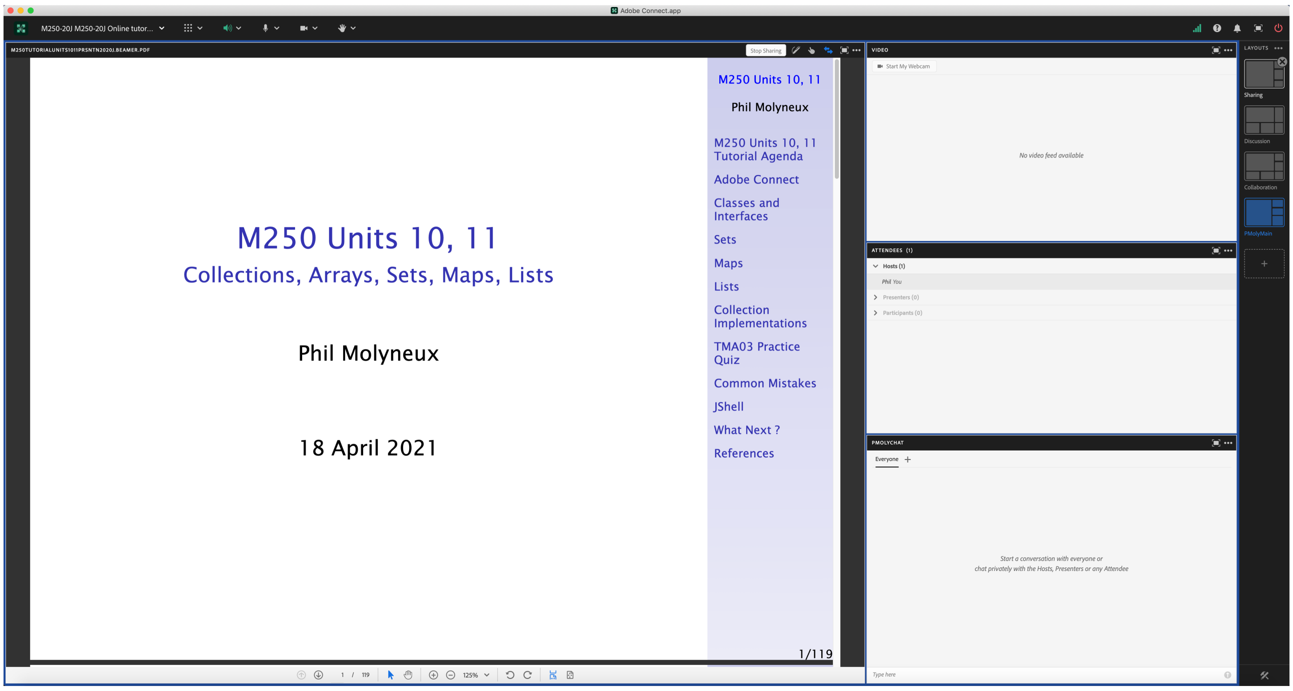
ToC

2 Adobe Connect Interface and Settings

2.1 Adobe Connect Interface

Adobe Connect Interface — Host View



M250 Units 10, 11
Collections, Arrays, Sets, Maps, Lists

Phil Molyneux

18 April 2021

M250 Units 10, 11
Phil Molyneux
M250 Units 10, 11 Tutorial Agenda
Adobe Connect
Classes and Interfaces
Sets
Maps
Lists
Collection Implementations
TMA03 Practice Quiz
Common Mistakes
JShell
What Next ?
References

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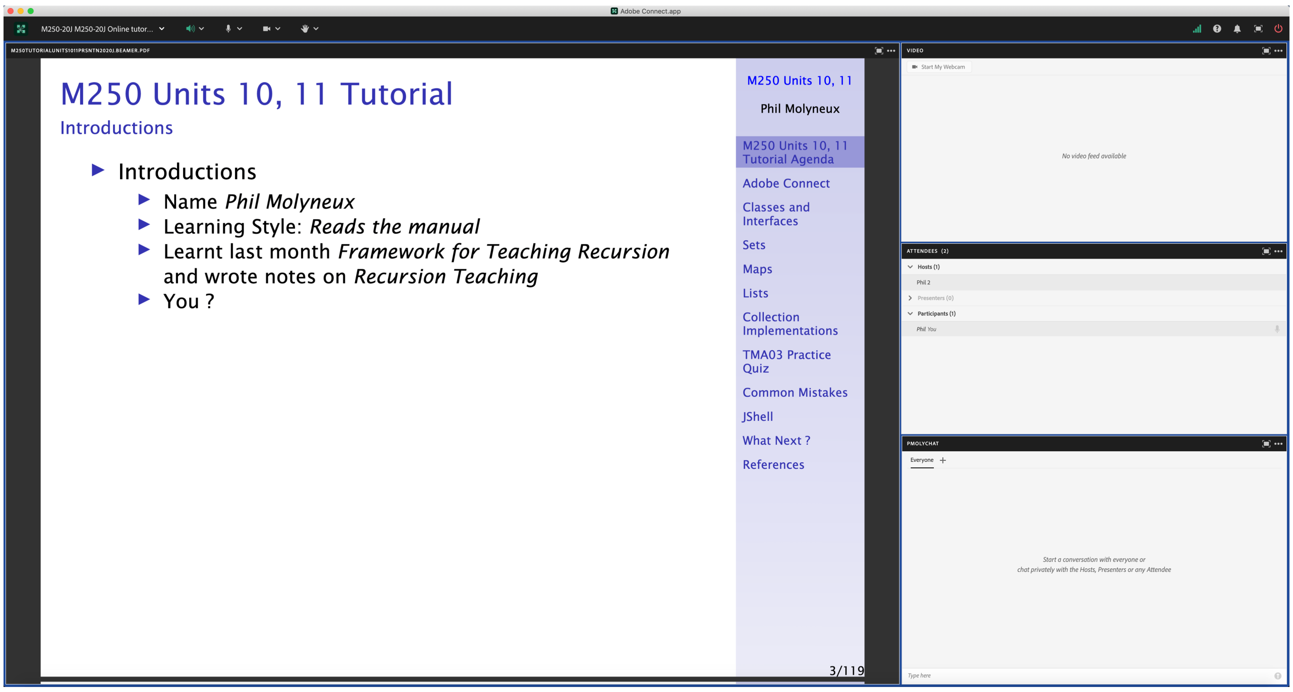
Adobe Connect Interface — Participant View



M250 Units 10, 11 Tutorial

Introductions

- Introductions
  - Name *Phil Molyneux*
  - Learning Style: *Reads the manual*
  - Learnt last month *Framework for Teaching Recursion* and wrote notes on *Recursion Teaching*
  - You ?

M250 Units 10, 11

Phil Molyneux

M250 Units 10, 11 Tutorial Agenda

Adobe Connect

Classes and Interfaces

Sets

Maps

Lists

Collection Implementations

TMA03 Practice Quiz

Common Mistakes

JShell

What Next ?

References

3/119

No video feed available

Start a conversation with everyone or chat privately with the Hosts, Presenters or any Attendee

2.2 Adobe Connect Settings

Adobe Connect — Settings

- **Everybody** *Menu bar* > *Meeting* > Speaker & Microphone Setup
- *Menu bar* > *Microphone* > Allow Participants to Use Microphone ✔
- Check Participants see the entire slide including slide numbers bottom right **Workaround**
  - *Disable Draw* *Share pod* > *Menu bar* > *Draw icon*
  - *Fit Width* *Share pod* > *Bottom bar* > *Fit Width icon* ✔
- *Meeting* > Preferences > General > **Host Cursor** > Show to all attendees
- *Menu bar* > *Video* > Enable **Webcam** for Participants ✔
- Do not *Enable single speaker mode*
- Cancel hand tool
- Do not enable green pointer
- **Recording** Meeting > Record Session ✔
- **Documents** Upload PDF with drag and drop to share pod
- Delete *Meeting* > Manage Meeting Information > Uploaded Content and *check filename* > *click on delete*

Adobe Connect — Access

- **Tutor Access**

  TutorHome > M269 Website > Tutorials

  Cluster Tutorials > M269 Online tutorial room

Tutor Groups › M269 Online tutor group room

Module-wide Tutorials › M269 Online module-wide room

- **Attendance**

  TutorHome › Students › View your tutorial timetables

- **Beamer Slide Scaling** 440% (422 x 563 mm)
- **Clear Everyone's Status**

  Attendee Pod › Menu › Clear Everyone's Status

- **Grant Access** and send link via email

  Meeting › Manage Access & Entry › Invite Participants. . .

- **Presenter Only Area**

  Meeting › Enable/Disable Presenter Only Area

Adobe Connect — Keystroke Shortcuts

- Keyboard shortcuts in Adobe Connect
- **Toggle Mic** ⌘+M (Mac), Ctrl+M (Win) (On/Disconnect)
- **Toggle Raise-Hand status** ⌘+E
- **Close dialog box** ⎋ (Mac), Esc (Win)
- **End meeting** ⌘+\

2.3 Adobe Connect — Sharing Screen & Applications

- Share My Screen › Application tab › Terminal for Terminal
- Share menu › Change View › Zoom in for mismatch of screen size/resolution (Participants)
- (Presenter) Change to 75% and back to 100% (solves participants with smaller screen image overlap)
- Leave the application on the original display
- Beware blued hatched rectangles — from other (hidden) windows or contextual menus
- Presenter screen pointer affects viewer display — beware of moving the pointer away from the application
- First time: System Preferences › Security & Privacy › Privacy › Accessibility

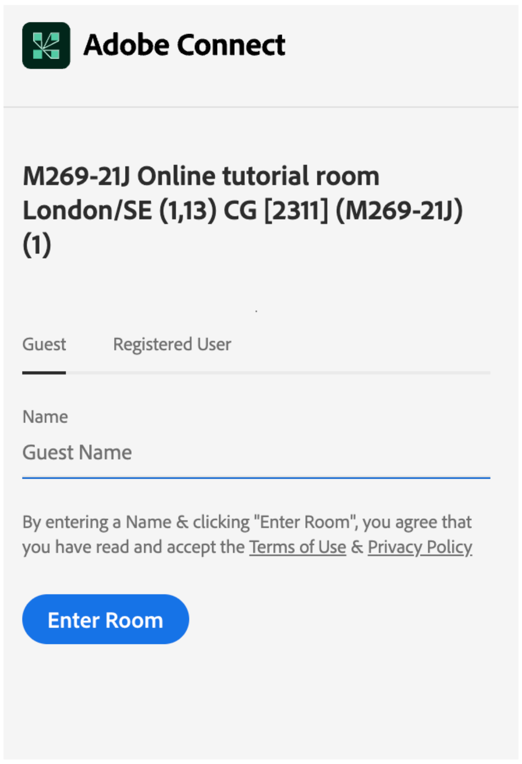
2.4 Adobe Connect — Ending a Meeting

- *Notes for the tutor only*
- **Student:** Meeting › Exit Adobe Connect
- **Tutor:**
- **Recording** Meeting › Stop Recording ✓

- **Remove Participants** Meeting › End Meeting... ✓
  - Dialog box allows for message with default message:
  - *The host has ended this meeting. Thank you for attending.*
- **Recording availability** *In course Web site for joining the room, click on the eye icon in the list of recordings under your recording* — edit description and name
- **Meeting Information** Meeting › Manage Meeting Information — can access a range of information in Web page.
- **Delete File Upload** Meeting › Manage Meeting Information › Uploaded Content tab select file(s) and click Delete
- **Attendance Report** see course Web site for joining room

2.5 Adobe Connect — Invite Attendees

- **Provide Meeting URL** Menu › Meeting › Manage Access & Entry › Invite Participants...
- **Allow Access without Dialog** Menu › Meeting › Manage Meeting Information provides new browser window with *Meeting Information* Tab bar › Edit Information
- Check *Anyone who has the URL for the meeting can enter the room*
- Default *Only registered users and accepted guests may enter the room*
- **Reverts to default next session but URL is fixed**
- Guests have blue icon top, registered participants have yellow icon top — same icon if URL is open
- See Start, attend, and manage Adobe Connect meetings and sessions
- Click on the link sent in email from the Host
- Get the following on a Web page
- As *Guest* enter your name and click on Enter Room


Adobe Connect

M269-21J Online tutorial room London/SE (1,13) CG [2311] (M269-21J) (1)

Guest Registered User

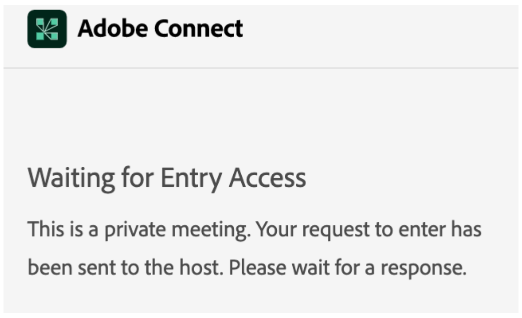
Name

Guest Name

By entering a Name & clicking "Enter Room", you agree that you have read and accept the Terms of Use & Privacy Policy

Enter Room

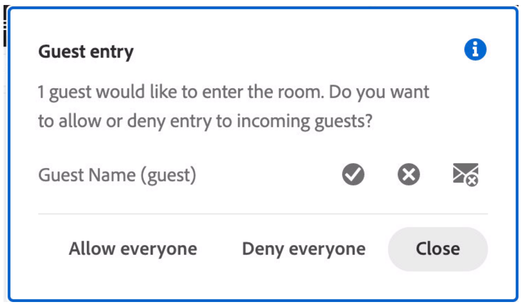
- See the *Waiting for Entry Access* for *Host* to give permission


Adobe Connect

Waiting for Entry Access

This is a private meeting. Your request to enter has been sent to the host. Please wait for a response.

- *Host* sees the following dialog in *Adobe Connect* and grants access


Guest entry

1 guest would like to enter the room. Do you want to allow or deny entry to incoming guests?

Guest Name (guest)

Allow everyone Deny everyone Close

2.6 Layouts

- **Creating new layouts** example *Sharing* layout
- Menu › Layouts › Create New Layout. . . Create a New Layout dialog › Create a new blank layout and name it *PMolyMain*
- New layout has no Pods but does have Layouts Bar open (see Layouts menu)
- **Pods**

- Menu › Pods › Share › Add New Share and resize/position — initial name is *Share n* — rename *PMolyShare*
- **Rename Pod** Menu › Pods › Manage Pods. . . Manage Pods › Select › Rename or Double-click & rename
- Add Video pod and resize/reposition
- Add Attendance pod and resize/reposition
- Add Chat pod — rename it *PMolyChat* — and resize/reposition
- Dimensions of **Sharing** layout (on 27-inch iMac)
  - Width of Video, Attendees, Chat column 14 cm
  - Height of Video pod 9 cm
  - Height of Attendees pod 12 cm
  - Height of Chat pod 8 cm
- **Duplicating Layouts** does *not* give new instances of the Pods and is probably not a good idea (apart from local use to avoid delay in reloading Pods)
- **Auxiliary Layouts** name *PMolyAux0n*
  - Create new Share pod
  - Use existing Chat pod
  - Use same Video and Attendance pods

2.7 Chat Pods

- **Format Chat text**
- Chat Pod › menu icon › My Chat Color
- Choices: Red, Orange, Green, Brown, Purple, Pink, Blue, Black
- Note: Color reverts to Black if you switch layouts
- Chat Pod › menu icon › Show Timestamps

2.8 Graphics Conversion for Web

- Conversion of the screen snaps for the installation of Anaconda on 1 May 2020
- Using GraphicConverter 11
- File › Convert & Modify › Conversion › Convert
- Select files to convert and destination folder
- Click on Start selected Function or ⌘ + ⏎

2.9 Adobe Connect Recordings

- *Menu bar* › *Meeting* › Preferences › Video
- Aspect ratio › Standard (4:3) (not Wide screen (16:9) default)

- Video quality › Full HD (1080p not High default 480p)
- **Recording** *Menu bar* › *Meeting* › Record Session ✓
- **Export Recording**
- *Menu bar* › *Meeting* › Manage Meeting Information
- *New window* › Recordings › *check Tutorial* › Access Type button
- check Public › check Allow viewers to download
- **Download Recording**
- *New window* › Recordings › *check Tutorial* › Actions › Download File

ToC

Commentary 2

2 Binary Trees

- Usage, terminology, example trees
- Representation, Abstract Data Types and notation
- Tree traversals, Depth First and Breadth First
- Recursive versions first
- Iterative versions derived from recursive versions
- Iterative depth first traversals for interest only
- Points on performance

ToC

3 Binary Trees — Introduction

- The *tree data structure* is the most widely used non-linear structure in many algorithms.
- Almost all algorithms that take logarithmic time, $O(\log n)$, do so because of an underlying tree structure.
- Common examples
- Binary search tree — this is used in many search applications
- Huffman coding tree — used in compression algorithms in, for example, JPEG and MP3 files
- Heaps — used to implement priority queues
- B-trees — generalisation of Binary search trees used in databases.

3.1 Binary Trees — Terminology

- **Binary Tree definition** — a Binary tree is either

  - an Empty Tree or
  - a Node with an item and two subtrees
  - One subtree is designated a left subtree and the other a right subtree
- Note that this is a recursive or inductive definition — this is very common in programming.
- Can also define trees as *graphs without cycles* — see *graph notes*

Other Recursive Data Structures

- Other examples of recursive or inductively defined data structures we have seen include:
- A List is either
  - an Empty List or
  - an Item followed by the rest of the list
- A Stack is either
  - an Empty Stack or
  - the Top item followed by the rest of the stack
- In each case the recursive nature of the data structure definition frequently gives a clue about how to write a recursive program for a computational problem.

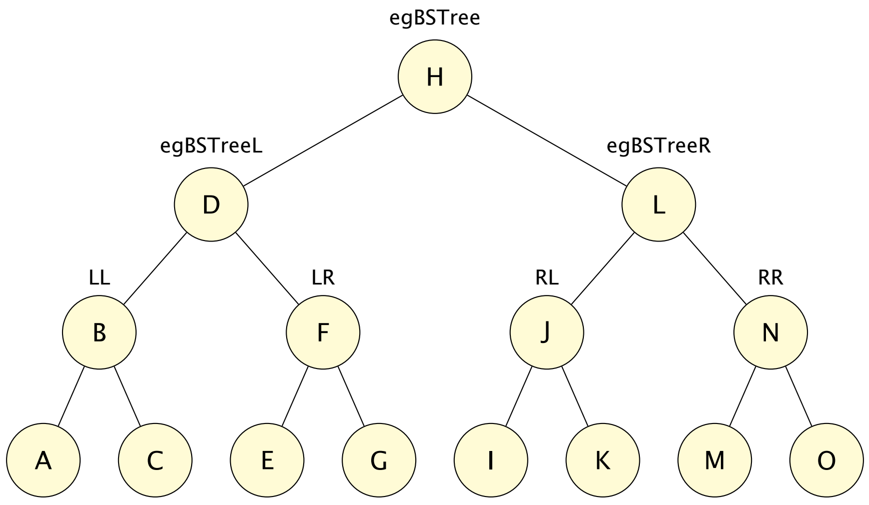
Binary Trees — Terminology

- Children — subtrees of a node that are not empty
- Leaves — nodes with two empty subtrees
- Full Binary Tree — every node other than the leaves has two non empty subtrees
- Perfect Binary Tree — all leaves are at the same level (or depth) children
- Complete Binary Tree — every level, except possibly the last, is completely filled, and all nodes are as far left as possible — used for *Binary Heap*
- **Health Warning:** the terminology varies from text to text and between graph theory in mathematics and computing.

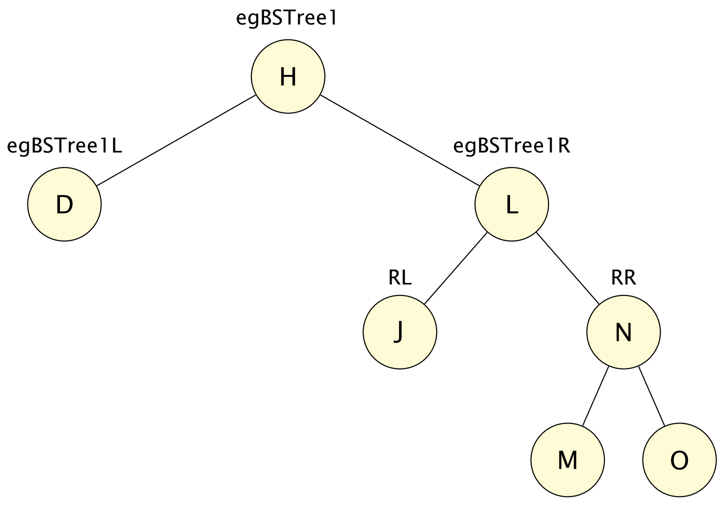
ToC

3.2 Binary Tree Examples

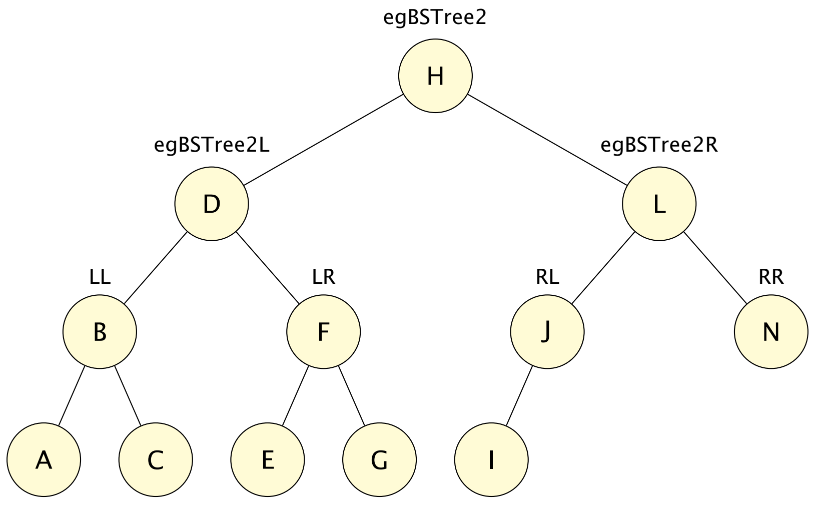
Example egBSTree



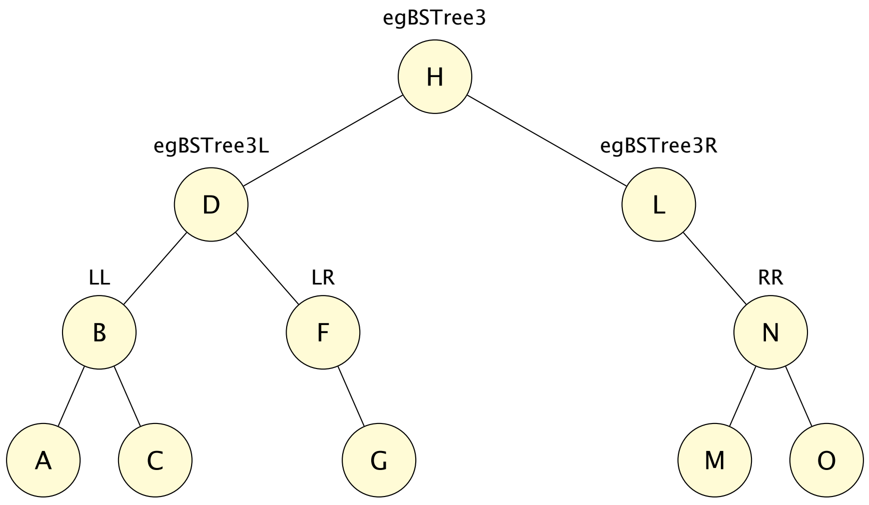
Example egBSTree1



Example egBSTree2



Example egBSTree3



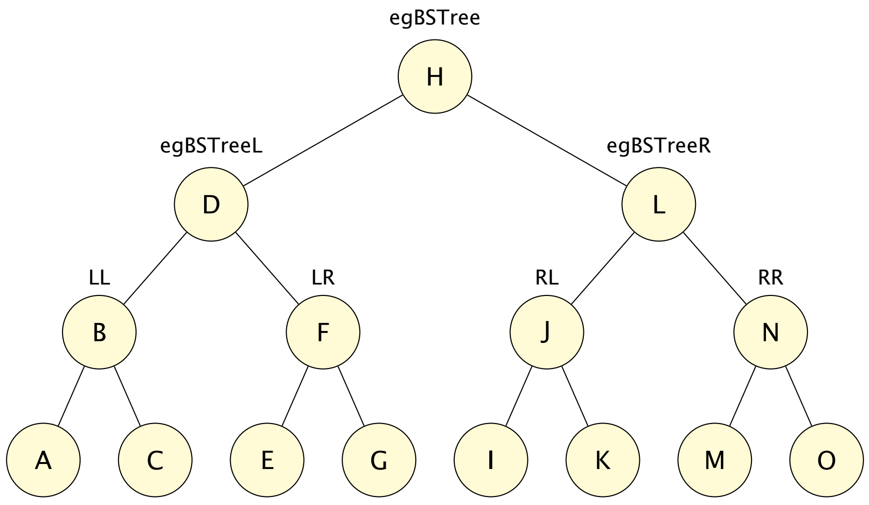
Activity 1 Binary Tree Types

- What types of trees are the above example trees ?
- egBSTree
- egBSTree1
- egBSTree2
- egBSTree3

Go to Answer

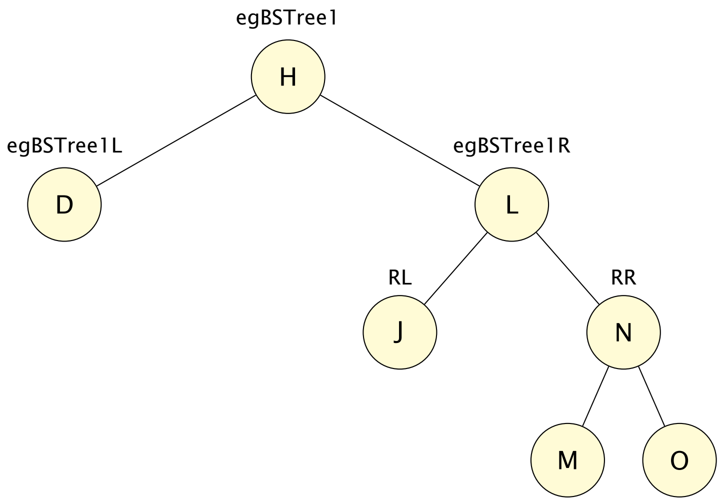
Answer 1 Binary Tree Types

- egBSTree — perfect



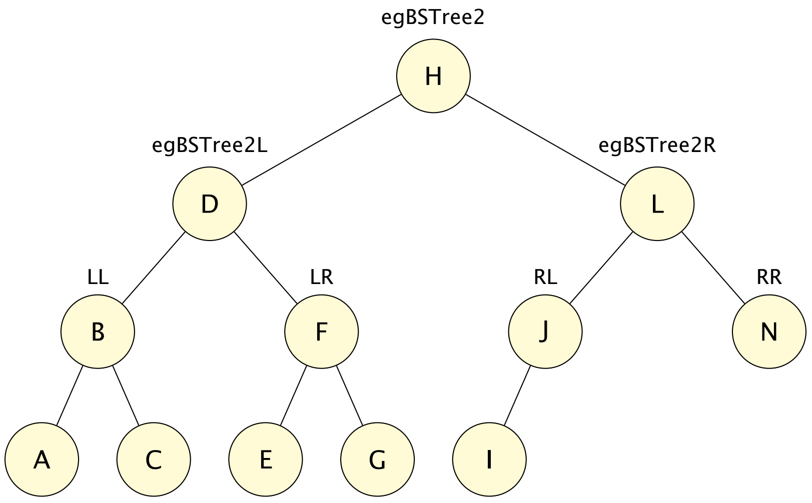
Answer 1 Binary Tree Types

- egBSTree1 — full



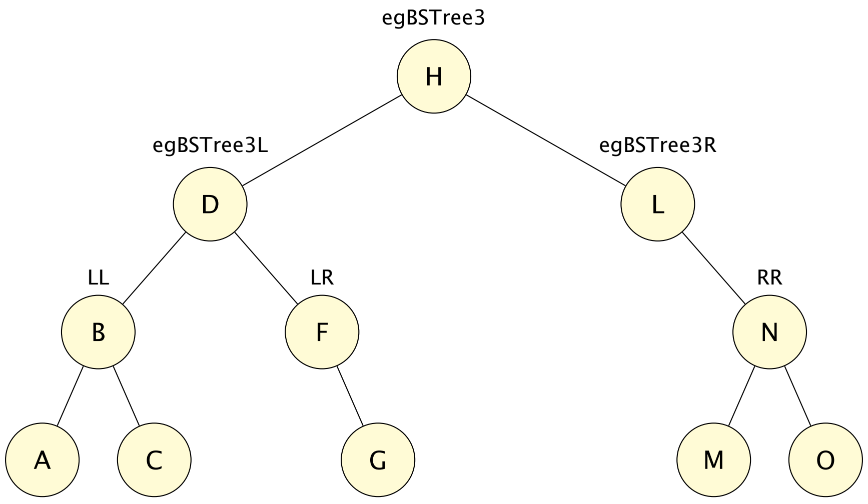
Answer 1 Binary Tree Types

- egBSTree2 — complete



Answer 1 Binary Tree Types

- egBSTree3 — just a binary tree



Go to Activity

ToC

3.3 Representation of Binary Trees

3.3.1 Python Representation from 2021J

- In 2021J M269 revision the Binary Tree Abstract Data Type (ADT) is represented by the following Python *Class*
- The code is in the M269 Jupyter Notebooks and the provided file `m269_trees.py`
- The code is reproduced in the file `M269BinaryTrees2021J.py` but, for brevity, without the docstrings

```python
class Tree :

  def __init__(self) :
    self.root = None
    self.left = None
    self.right = None

def is_empty(tree: Tree) -> bool :
  return (tree.root == tree.left == tree.right
          == None)

def join(item: object, left: Tree, right: Tree) -> Tree :
  tree = Tree()
  tree.root = item
  tree.left = left
  tree.right  = right
  return tree
```

- The functions `is_leaf`, `size`, `height`

```python
def is_leaf(tree: Tree) -> bool :
  return (not is_empty(tree)
          and is_empty(tree.left) and is_empty(tree.right))

def size(tree: Tree) -> int :
  if is_empty(tree) :
    return 0
  else :
    return (size(tree.left) + size(tree.right) + 1)

def height(tree: Tree) -> int :
  if is_empty(tree) :
    return 0
  else :
    return (max(height(tree.left), height(tree.right)) + 1)
```

- This representation works (see the M269 book) but has the slight disadvantage in the it has no default print representation that is useful
- For example, here is what happens when we attempt to print a very small tree — `threeBT` is the same as `THREE` in the chapter

```
Python3>>> threeBT = join(3,Tree(),Tree())
Python3>>> threeBT
<M269BinaryTrees2021J.Tree object at 0x10095a0a0>
Python3>>>
```

- We *could* write a method to implement a print representation of an instance of the class `Tree()` but that might be a lot of code
- The main point of having an Abstract Data Type (ADT) is we can swap out the underlying implementation for another one
- This might be done for efficiency reasons but here we do it to get an underlying type with a default print representation

- All we have to do is keep operations which provide access to the underlying representation
- *This is called a learning opportunity!*

ToC

3.3.2 Python Alternate Representation

- We first list the operations which will (or might) need to have direct access to the underlying representation
- Make an empty tree
- Construct a new tree from an item and two trees
- Query if a given tree is an empty tree
- Given a tree, return the left sub tree
- Given a tree, return the right sub tree
- Find the height of a tree
- Find the size of a tree
- The last two do not *need* access to the underlying representation (we can calculate the size and height with just the other operations) but as we will see later, we might give access for efficiency reasons
- To fully hide the ADT implementation we give common function names to the operations

Tree Class	**Common Name**	**Category**
`Tree()`	`mkEmptyBT()`	Constructor
`join()`	`mkNodeBT()`	Constructor
`is_empty()`	`isEmptyBT()`	Inspector
`tree.root`	`getDataBT()`	Destructor
`tree.left`	`getLeftBT()`	Destructor
`tree.right`	`getRightBT()`	Destructor
`height()`	`heightBT()`	Operation
`size()`	`sizeBT()`	Operation

- The functions labelled Constructor, Inspector, Destructor are operations that have direct access to the underlying representation
- `sizeBT()` and `heightBT()` are just ordinary operations in this version of the Tree ADT but for efficiency reasons they may become Inspectors in a later version
- We shall represent nodes by a named tuple — a *quick and dirty* object recommended by Guido van Rossum (author of the Python programming language).
- `namedtuple()` is a factory function for creating tuple subclasses with named fields
- It is imported from the `collections` module.

- It has a default print representation

```
7  from collections import namedtuple

9  EmptyBT = namedtuple('EmptyBT',[])

11 NodeBT = namedtuple('NodeBT'
12                     ,['dataBT','leftBT','rightBT'])
```

- The Python code above is in the file Python/M269TutorialBinaryTrees2022.py
- The line numbers in the margin correspond to the line numbers in the file.
- Notational convention:
- Python reserved identifiers are shown in **this color**
- Python buit-in functions in this color
- User defined data constructors and functions such as NodeBT and EmptyBT are shown in that color
- **Health Warning:** these notes may not be totally consistent with syntax colouring.
- We declare the Python type for a *Union* type since a Tree is either an empty tree or a non-empty tree
- This is venturing into some of the areas of Python Type Annotations that feel rather awkward but we shall use them in a simple way
- Remember that the Python interpreter only checks the type annotations for validity but not for correctness — they just have to look like proper types but the processor does *not* check them

```
14 # Tree type

16 from typing import TypeVar,Union,NewType

18 T = TypeVar('T')
19 Tree = NewType('Tree',Union[EmptyBT,NodeBT]).
```

- Note that using namedtuple means that all items are assumed to have Any types (see mypy: Named tuples)
- You *could* use NamedTuple which is a typed version of namedtuple but this would be getting a lot more complicated than types as used in M269
- in particular you would get involved in specifying user-defined generic types and forward references (since the Tree data type is recursive)
- We could have avoided Union by just having NodeBT and representing an empty tree by the Python None
- This would be isomorphic to the Class version with default printing

ToC

3.3.3 Operations

- We now provide functions to create, inspect and take apart binary trees
- The code with the line numbers is the code for the implementation using namedtuples

```
def mkEmptyBT() -> Tree :
  return EmptyBT()

def mkNodeBT(x : T,t1 : Tree,t2 : Tree) -> Tree :
  return NodeBT(x,t1,t2)

def isEmptyBT(t : Tree) -> bool:
  return t == EmptyBT()
```

- mkEmptyBT, mkNodeBT are *constructor* functions — we could have used the raw named tuples but the discipline is good for you and it makes it easier to refactor in future
- isEmptyBT uses the == operator for identity check (not identity (is))
- this is in line with *PEP 8 — Style Guide for Python Code*
- to see an example of why using (==) for comparison with None may produce odd results see When is the == operator not equivalent to the is operator? or Python None comparison: shouldI use is or ==?
- The code with no line numbers illustrates how the previous implementation using Class Tree can be given the same operations interface

```
def mkEmptyBT() -> Tree :
  return Tree()

def mkNodeBT(x : T,t1 : Tree,t2 : Tree) -> Tree :
  return join(x,t1,t2)

def isEmptyBT(t : Tree) -> bool:
  return is_empty(t)
```

- Here are the operations that access the parts of the tree

```
def getDataBT(t : Tree) -> T:
  if isEmptyBT(t):
    raise RuntimeError("getDataBT␣applied␣to␣EmptyBT()")
  else:
    return t.dataBT

def getLeftBT(t : Tree) -> Tree :
  if isEmptyBT(t):
    raise RuntimeError("getLeftBT␣applied␣to␣EmptyBT()")
  else:
    return t.leftBT

def getRightBT(t : Tree) -> Tree :
  if isEmptyBT(t):
    raise RuntimeError("getRightBT␣applied␣to␣EmptyBT()")
  else:
    return t.rightBT
```

- The Class Tree implementation of the above operations

```
def getDataBT(t : Tree) -> T:
  if isEmptyBT(t):
    raise RuntimeError("getDataBT␣applied␣to␣empty␣tree")
  else:
    return t.root

def getLeftBT(t : Tree) -> Tree :
  if isEmptyBT(t):
    raise RuntimeError("getLeftBT␣applied␣to␣empty␣tree")
  else:
    return t.left

def getRightBT(t : Tree) -> Tree :
  if isEmptyBT(t):
    raise RuntimeError("getRightBT␣applied␣to␣empty␣tree")
  else:
    return t.right
```

- Here are the operations heightBT() and sizeBT()
- Note that height of an empty tree is 0

```
def heightBT(t : Tree) -> int :
  if isEmptyBT(t):
    return 0
  else:
    return (1 + max(heightBT(getLeftBT(t))
                   ,heightBT(getRightBT(t))))

def sizeBT(t : Tree) -> int :
  if isEmptyBT(t) :
    return 0
  else :
    return (1 + sizeBT(getLeftBT(t))
              + sizeBT(getRightBT(t)))
```

- The Class Tree implementation of the above operations is exactly the same
- If we make height or size directly part of the data structure this may change

ToC

Activity 2 Python Representation

- Write Python implementations of the following trees (from the diagrams above) using the named tuple NodeBT and EmptyBT
- egBSTree
- egBSTree1
- egBSTree2
- egBSTree3

Go to Answer

Answer 2 Python Representation — egBSTree

```
  egBSTree = mkNodeBT('H',
               mkNodeBT('D',
                 mkNodeBT('B',
                   mkNodeBT('A',mkEmptyBT(),mkEmptyBT()),
                   mkNodeBT('C',mkEmptyBT(),mkEmptyBT())
                 ),
                 mkNodeBT('F',
                   mkNodeBT('E',mkEmptyBT(),mkEmptyBT()),
                   mkNodeBT('G',mkEmptyBT(),mkEmptyBT())
                 )
               ),
               mkNodeBT('L',
                 mkNodeBT('J',
                   mkNodeBT('I',mkEmptyBT(),mkEmptyBT()),
                   mkNodeBT('K',mkEmptyBT(),mkEmptyBT())
                 ),
                 mkNodeBT('N',
                   mkNodeBT('M',mkEmptyBT(),mkEmptyBT()),
                   mkNodeBT('O',mkEmptyBT(),mkEmptyBT())
                 )
               )
             )
```

Answer 2 Python Representation — egBSTree1

```
egBSTree1 = mkNodeBT('H',
               mkNodeBT('D',mkEmptyBT(),mkEmptyBT()),
               mkNodeBT('L',
                 mkNodeBT('J',mkEmptyBT(),mkEmptyBT()),
                 mkNodeBT('N',
                   mkNodeBT('M',mkEmptyBT(),mkEmptyBT()),
                   mkNodeBT('O',mkEmptyBT(),mkEmptyBT())
                 )
               )
             )
```

Answer 2 Python Representation — egBSTree2

```
egBSTree2 = mkNodeBT('H',
               mkNodeBT('D',
                 mkNodeBT('B',
                   mkNodeBT('A',mkEmptyBT(),mkEmptyBT()),
                   mkNodeBT('C',mkEmptyBT(),mkEmptyBT())
                 ),
                 mkNodeBT('F',
                   mkNodeBT('E',mkEmptyBT(),mkEmptyBT()),
                   mkNodeBT('G',mkEmptyBT(),mkEmptyBT())
                 )
               ),
               mkNodeBT('L',
                 mkNodeBT('J',
                   mkNodeBT('I',mkEmptyBT(),mkEmptyBT()),
                   mkEmptyBT()
                 ),
                 mkNodeBT('N',mkEmptyBT(),mkEmptyBT())
               )
             )
```

Answer 2 Python Representation — egBSTree3

```
egBSTree3 = mkNodeBT('H',
               mkNodeBT('D',
                 mkNodeBT('B',
                   mkNodeBT('A',mkEmptyBT(),mkEmptyBT()),
                   mkNodeBT('C',mkEmptyBT(),mkEmptyBT())
                 ),
                 mkNodeBT('F',
                   mkEmptyBT(),
                   mkNodeBT('G',mkEmptyBT(),mkEmptyBT())
                 )
               ),
               mkNodeBT('L',
                 mkEmptyBT(),
                 mkNodeBT('N',
                   mkNodeBT('M',mkEmptyBT(),mkEmptyBT()),
                   mkNodeBT('O',mkEmptyBT(),mkEmptyBT())
                 )
               )
             )
```

Go to Activity

Answer 2 Python Representation — egBSTreeL

```
egBSTreeL = mkNodeBT('D',
              mkNodeBT('B',
                mkNodeBT('A',mkEmptyBT(),mkEmptyBT()),
                mkNodeBT('C',mkEmptyBT(),mkEmptyBT())
              ),
              mkNodeBT('F',
                mkNodeBT('E',mkEmptyBT(),mkEmptyBT()),
                mkNodeBT('G',mkEmptyBT(),mkEmptyBT())
              )
            )
```

Go to Activity

Answer 2 Python Representation — egBSTreeLL

```
146 egBSTreeLL = mkNodeBT('B',
147                 mkNodeBT('A',mkEmptyBT(),mkEmptyBT()),
148                 mkNodeBT('C',mkEmptyBT(),mkEmptyBT())
149               )
```

Go to Activity

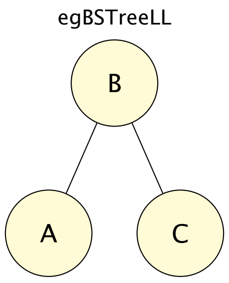
3.4 Binary Tree Traversals

- Many applications require visiting each node in a binary tree and doing some processing.
- This could be adding quantities to find a total, identifying the number of nodes with a particular property and so on.
- There are several common patterns of visiting each node or traversing a tree
  - Depth first where the search tree is deepened as much as possible on each child before visiting the next sibling
  - Breadth first where we visit every node on a level before visiting the next level
- Each traversal takes a tree and returns a list of items at the nodes of the tree

ToC

3.5 Tree Traversals — Depth First

- **In-Order traversal of tree `t`**
  1. If `t` is an empty tree then return the empty list
  2. Otherwise do an In Order traversal of the left subtree of `t` then append a list just containing the data item at the root of `t` followed by an In Order traversal of the right subtree of `t`
- **Pre-Order traversal of tree `t`**
  - As In-Order but output a list with the item at the root of `t` before traversing the two subtrees
- **Post-Order traversal of tree `t`**
  - As Pre-Order but output a list with the item at the root of `t` after traversing the two subtrees
- *In-Order*, *Pre-Order* and *Post-Order* traversals are collectively termed *Depth First Traversals*
- We first provide the usual recursive implementations — in a later section we translate the recursive versions into iterative versions
- Tree egBSTreeLL Python code at line 146 on page 22



- The *Depth first* traversals are implemented in Python by inOrderBT(), preOrderBT() and postOrderBT()

```
Python3>>> inOrderBT(egBSTreeLL)
['A', 'B', 'C']
Python3>>> preOrderBT(egBSTreeLL)
['B', 'A', 'C']
Python3>>> postOrderBT(egBSTreeLL)
['A', 'C', 'B']
```

```
311 def inOrderBT(t) :
312   if isEmptyBT(t) :
313     return []
314   else :
315     return (inOrderBT(getLeftBT(t)) + [getDataBT(t)]
316             + inOrderBT(getRightBT(t)))

318 def preOrderBT(t) :
319   if isEmptyBT(t) :
320     return []
321   else :
322     return ([getDataBT(t)] + preOrderBT(getLeftBT(t))
323             + preOrderBT(getRightBT(t)))

325 def postOrderBT(t) :
326   if isEmptyBT(t) :
327     return []
328   else :
329     return (postOrderBT(getLeftBT(t))
330             + postOrderBT(getRightBT(t)) + [getDataBT(t)])
```

Activity 3 Depth First Traversals

- Give the lists of items in an in-order traversal of egBSTree, egBSTree1, egBSTree2, egBSTree3
- Give the lists of items in a pre-order traversal of egBSTree, egBSTree1, egBSTree2, egBSTree3
- Give the lists of items in a post-order traversal of egBSTree, egBSTree1, egBSTree2, egBSTree3
- Depth first traversal code is from line 311 on page 23 (Python)
- Binary tree code is from line 70 on page 20 (Python)

Ans 3

Answer 3 Depth First Traversals — In-Order

```
Python3>>> inOrderBT(egBSTree)
['A', 'B', 'C', 'D', 'E', 'F', 'G',
 'H', 'I', 'J', 'K', 'L', 'M', 'N', 'O']
Python3>>> inOrderBT(egBSTree1)
['D', 'H', 'J', 'L', 'M', 'N', 'O']
Python3>>> inOrderBT(egBSTree2)
['A', 'B', 'C', 'D', 'E', 'F', 'G',
 'H', 'I', 'J', 'L', 'N']
Python3>>> inOrderBT(egBSTree3)
```

```
['A', 'B', 'C', 'D', 'F', 'G', 'H',
 'L', 'M', 'N', 'O']
```

- (Line breaks introduced for layout)

Answer 3 Depth First Traversals — Pre-Order

```
Python3>>> preOrderBT(egBSTree)
['H', 'D', 'B', 'A', 'C', 'F', 'E',
 'G', 'L', 'J', 'I', 'K', 'N', 'M', 'O']
Python3>>> preOrderBT(egBSTree1)
['H', 'D', 'L', 'J', 'N', 'M', 'O']
Python3>>> preOrderBT(egBSTree2)
['H', 'D', 'B', 'A', 'C', 'F', 'E',
 'G', 'L', 'J', 'I', 'N']
Python3>>> preOrderBT(egBSTree3)
['H', 'D', 'B', 'A', 'C', 'F', 'G',
 'L', 'N', 'M', 'O']
```

Answer 3 Depth First Traversals — Post-Order

```
Python3>>> postOrderBT(egBSTree)
['A', 'C', 'B', 'E', 'G', 'F', 'D',
 'I', 'K', 'J', 'M', 'O', 'N', 'L', 'H']
Python3>>> postOrderBT(egBSTree1)
['D', 'J', 'M', 'O', 'N', 'L', 'H']
Python3>>> postOrderBT(egBSTree2)
['A', 'C', 'B', 'E', 'G', 'F', 'D',
 'I', 'J', 'N', 'L', 'H']
Python3>>> postOrderBT(egBSTree3)
['A', 'C', 'B', 'G', 'F', 'D', 'M',
 'O', 'N', 'L', 'H']
```

Act 3

ToC

3.6 Tree Traversals — Breadth First

- The M269 book section 16.3.5 gives an iterative version of a breadth first traversal but only mentions a recursive version briefly
- We shall start with a recursive version and transform that by stages into the iterative version in the book
- I find it easier to think of the recursive version first — you should observe how people think they think about programming
- First we do some exercises

Activity 4 Breadth First Traversals

- A *level order* traversal of a binary tree takes a tree and returns the list of levels
- Each level is the list of items at that level
- Give the list of levels for:
- egBSTree
- egBSTreeL
- egBSTree1
- egBSTree2

- egBSTree3

Go to Answer

Answer 4 Breadth First Traversals

- Answer 4 Breadth First Traversals

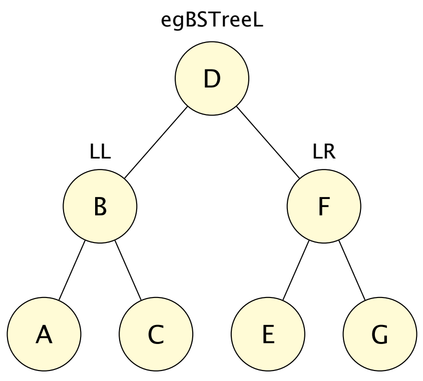
```
Python3>>> levelOrderBT(egBSTree)
[['H'], ['D', 'L'], ['B', 'F', 'J', 'N'],
  ['A', 'C', 'E', 'G', 'I', 'K', 'M', 'O']]
Python3>>> levelOrderBT(egBSTreeL)
[['D'], ['B', 'F'], ['A', 'C', 'E', 'G']]
Python3>>> levelOrderBT(egBSTree1)
[['H'], ['D', 'L'], ['J', 'N'], ['M', 'O']]
Python3>>> levelOrderBT(egBSTree2)
[['H'], ['D', 'L'], ['B', 'F', 'J', 'N'],
  ['A', 'C', 'E', 'G', 'I']]
Python3>>> levelOrderBT(egBSTree3)
[['H'], ['D', 'L'], ['B', 'F', 'N'], ['A', 'C', 'G', 'M', 'O']]
Python3>>>
```

Go to Activity

Answer 4 Breadth First Traversals

- Answer 4 Breadth First Traversals

```
Python3>>> levelOrderBT(egBSTreeL)
[['D'], ['B', 'F'], ['A', 'C', 'E', 'G']]
```



3.6.1 Breadth First V01

- The first version will be recursive and driven by the structure of trees and will start by writing `levelOrder()` which takes a binary tree and returns a list of levels — a `level` is a list of items at the level

(1) An empty tree has an empty list of levels (level zero)

(2) A non-empty tree has the list of the root item followed by combining the two lists of the levels for the two sub-trees

- We will call the function that combines the two lists of levels `longZipMerge()` since it is similar to the Python library `zip()` function

```
def levelOrderBT(t : Tree) -> [[T]] :
  if isEmptyBT(t) :
    return []
  else :
    x = getDataBT(t)
    left = getLeftBT(t)
    right = getRightBT(t)
    return ([[x]] +
436        longZipMerge(levelOrderBT(left),levelOrderBT(right)))
```

- longZipMerge() is a variant on the Python library function zip()
- zip() iterates over several iterables in parallel, producing tuples with an item from each one.
- longZipMerge() takes two lists and returns a new list with merged pairs of items from each list which is a level order traverse of the subtrees
- The two lists do not need to be of the same length — any excess is just appended to the merged result so far

```
438   def longZipMerge(xss : [[T]],yss : [[T]]) -> [[T]] :
439     if xss == [] :
440       return yss
441     elif yss == [] :
442       return xss
443     else :
444       return ([xss[0] + yss[0]] + longZipMerge(xss[1:],yss[1:]))
```

- Evaluation of levelOrderBT(egBSTreeL)

```
levelOrderBT(egBSTreeLL)
= [['B']]                         # by line 431
   + longZipMerge(levelOrderBT(egBSTreeLLL),
                  levelOrderBT(egBSTreeLLR))
= [['B']]                         # by lines 431,439
   + longZipMerge([['A']],[['C']])
= [['B']] + [['A','C']]           # by line 441
= [['B'],['A','C']]
```

```
levelOrderBT(egBSTreeLR)
= [['F'],['E','G']]               # as above
```

```
levelOrderBT(egBSTreeL)
= [['D']]                         # by line 431
   + longZipMerge(levelOrderBT(egBSTreeLL),
                  levelOrderBT(egBSTreeLR))
= [['D']]                         # as above
   + longZipMerge([['B'],['A','C']],
                  [['F'],['E','G']])
= [['D']]                         # by line 441
   + [['B','F'],['A','C','E','G']]
= [['D'],['B','F'],['A','C','E','G']]  # correct - check the steps
```

- Testing can only show the presence of bugs but not the absence of bugs (Edsger W Dijkstra Quotes)
- We shall now investigate a similar program with a subtle error

```
465   def levelOrderBT01(t : Tree) -> [[T]] :
466     if isEmptyBT(t) :
467       return []
468     else :
469       x = getDataBT(t)
470       left = getLeftBT(t)
471       right = getRightBT(t)
472       return ([x] +
473         longZipMerge01(levelOrderBT01(left),levelOrderBT01(right)))

475   def longZipMerge01(xss : [[T]],yss : [[T]]) -> [[T]] :
476     if xss == [] :
477       return yss
478     elif yss == [] :
479       return xss
480     else :
481       return ([xss[0],yss[0]] + longZipMerge01(xss[1:],yss[1:]))
```

- We first do a few tests

```
Python3>>> levelOrderBT(egBSTreeLL)
[['B'], ['A', 'C']]
Python3>>> levelOrderBT01(egBSTreeLL)
['B', 'A', 'C']
Python3>>> levelOrderBT(egBSTree1)
[['H'], ['D', 'L'], ['J', 'N'], ['M', 'O']]
Python3>>> levelOrderBT01(egBSTree1)
['H', 'D', 'L', 'J', 'N', 'M', 'O']
```

- Correct order but a list of items not a list of levels

```
Python3>>> levelOrderBT(egBSTreeL)
[['D'], ['B', 'F'], ['A', 'C', 'E', 'G']]
Python3>>> levelOrderBT01(egBSTreeL)
['D', 'B', 'F', 'A', 'E', 'C', 'G']
```

- Wrong order — we now do an evaluation to see where the error is
- Evaluation of `levelOrderBT01(egBSTreeL)`

```
levelOrderBT01(egBSTreeLL)
= ['B']                               # by line 468
   + longZipMerge01(levelOrderBT01(egBSTreeLLL),
                    levelOrderBT01(egBSTreeLLR))
= ['B']                               # by lines 468,476
   + longZipMerge01(['A'],['C'])
= ['B'] + ['A','C']                   # by line 478
= ['B','A','C']
```

```
levelOrderBT01(egBSTreeLR)
= ['F','E','G']                       # as above
```

```
levelOrderBT01(egBSTreeL)
= ['D']                               # by line 468
   + longZipMerge01(levelOrderBT01(egBSTreeLL),
                    levelOrderBT01(egBSTreeLR)))
= ['D']                               # as above
   + longZipMerge01(['B','A','C'],
                    ['F','E','G'])
= ['D']                               # by line 478
   + ['B','F','A','E','C','G']
= ['D','B','F','A','E','C','G']       # notice the error ?
```

- `levelOrderBT01()` is not only of the wrong type but produces the wrong order except for a limited number of trees
- The Python type annotations are only checked for syntax but not for correctness
- We get the final `breadthBT01()` by flattening the list of levels
- This uses a *list comprehension* as a shorthand for nested loops — see explanation below

```
454  def flattenLevels(levels : [[T]]) -> [T] :
455    return ([elem for level in levels for elem in level])

457  def breadthBT01(t : Tree) -> [T] :
458    return flattenLevels(levelOrderBT(t))
```

- Python List comprehensions (tutorial), List comprehensions (reference) — a neat way of expressing iterations over a list
- Example (a) Square the even numbers between 0 and 9
- Example (b) Generate a list of pairs which satisfy some condition

```
Python3>>> [x ** 2 for x in range(10) if x % 2 == 0]
[0, 4, 16, 36, 64]
Python3>>> [(x,y) for x in range(4)
...               for y in range(4)
...               if x % 2 == 0
...                  and y % 3 == 0]
[(0, 0), (0, 3), (2, 0), (2, 3)]
Python3>>>
```

- In general

```
  [expr for target1 in iterable1 if cond1
        for target2 in iterable2 if cond2 ...
        for targetN in iterableN if condN ]
```

- Instead of the list comprehension, `flattenLevels()` could be defined with an accumulating list and nested loops.
- M269 does not mention list comprehensions so you would have to decide whether they are worth mentioning

```
def flattenLevelsA(levels : [[T]]) -> [T] :
  accumList = []
  for level in levels :
    for elem in level :
      accumList = accumList + [elem]
  return accumList
```

ToC

3.6.2 Breadth First V02

- We now have a correct program `breadthBT01()` but this does lots of (how many?) traversals of the data — we may want a more efficient version and hence we transform our program
- Version 02 uses a helper function `bfTraverse(vs,ts)` which takes a list of item seen, `vs`, and a list (or queue) of trees to be visited, `ts`
- As we visit a node, we add its subtree to the queue, `ts`

```
def breadthBT02(t : Tree) -> [T] :
  return bfTraverse([],[t])

def bfTraverse(vs : [T],ts : [Tree]) -> [T] :
  if ts == []:
    return vs
  elif isEmptyBT(ts[0]):
    return bfTraverse(vs,ts[1:])
  else:
    return (bfTraverse(vs + [getDataBT(ts[0])],
                ts[1:] + [getLeftBT(ts[0]),getRightBT(ts[0])]))
```

ToC

3.6.3 Breadth First V03

- Version 02 removed some of the recursion by enqueueing trees to be visited
- This version has the disadvantage that no output until all the nodes are visited, which could mean a long wait or never if the tree is infinite

- Version 03 enables a lazier approach — Python could use a Generator expression or augment the code with Yield expressions (both not used in M269) but other languages, such as Haskell use lazy evaluation by default

```python
def breadthBT03(t : Tree) -> [T] :
  return lbfBT([t])

def lbfBT(ts : [Tree]) -> [T] :
  if ts == []:
    return []
  elif isEmptyBT(ts[0]):
    return lbfBT(ts[1:])
  else:
    return ([getDataBT(ts[0])]
        + lbfBT(ts[1:] + [getLeftBT(ts[0]),getRightBT(ts[0])]))
```

3.6.4 Breadth First V04

- Version 03 has the only recursive call as (almost) the last thing
- So we can implement this with a `while` loop

```python
def breadthBT04(t : Tree) -> [T] :
  ts = [t] # Trees to visit
  vs = []  # Values seen
  while (ts != []) :
    if not (isEmptyBT(ts[0])) :
      vs = vs + [getDataBT(ts[0])]
      ts = ts[1:] + [getLeftBT(ts[0]),getRightBT(ts[0])]
    else :
      ts = ts[1:]
  return vs
```

Activity 5 Breadth First Error

- There is an error in the following version — what is the error ?
- Why would the print statements not help ?
- Why don’t the Python type annotations help ?

```python
543 def breadthBT04A(t : Tree) -> [T] :
544   ts = [t] # Trees to visit
545   vs = []  # Values seen
546   while not (isEmptyBT(ts)) :
547     print('ts␣=␣',ts)
548     if not (isEmptyBT(ts[0])) :
549       print('len(ts)␣=␣',len(ts))
550       print('getDataBT(ts[0])␣=␣',getDataBT(ts[0]))
551       vs = vs + [getDataBT(ts[0])]
552       ts = ts[1:] + [getLeftBT(ts[0]),getRightBT(ts[0])]
553     else :
554       print('ts[0]␣is␣empty','len(ts)␣=␣',len(ts))
555       ts = ts[1:]
556   return vs
```

Go to Answer

Answer 5 Breadth First Error

- The error is the `while` condition at line 546
- `isEmptyBT(ts)` will never return `True` since `ts` is a list of trees
- The article version of these notes contains an output of the print statements and the error report
- The error is reported at line 548 as *IndexError: list index out of range*

- So the print statements do not show the real error
- The Python interpreter does not check the type annotations for correctness, just the syntax
- Remember that Python is a weakly typed language

Go to Activity

- Here is a sample error listing for breadthBT04A()

```
Python3>>> breadthBT04(egBSTreeL)
['D', 'B', 'F', 'A', 'C', 'E', 'G']
Python3>>> breadthBT04A(egBSTreeL)
ts =  [NodeBT(dataBT='D', leftBT=NodeBT(dataBT='B', leftBT=NodeBT(dataBT='A', leftBT=EmptyBT(),  ↙
    ↗ rightBT=EmptyBT()), rightBT=NodeBT(dataBT='C', leftBT=EmptyBT(), rightBT=EmptyBT())),  ↙
    ↗ rightBT=NodeBT(dataBT='F', leftBT=NodeBT(dataBT='E', leftBT=EmptyBT(), rightBT=EmptyBT()),  ↙
    ↗ rightBT=NodeBT(dataBT='G', leftBT=EmptyBT(), rightBT=EmptyBT())))]
len(ts) =  1
getDataBT(ts[0]) =  D
ts =  [NodeBT(dataBT='B', leftBT=NodeBT(dataBT='A', leftBT=EmptyBT(), rightBT=EmptyBT()),  ↙
    ↗ rightBT=NodeBT(dataBT='C', leftBT=EmptyBT(), rightBT=EmptyBT())), NodeBT(dataBT='F',  ↙
    ↗ leftBT=NodeBT(dataBT='E', leftBT=EmptyBT(), rightBT=EmptyBT()), rightBT=NodeBT(dataBT='G',  ↙
    ↗ leftBT=EmptyBT(), rightBT=EmptyBT()))]
len(ts) =  2
getDataBT(ts[0]) =  B
ts =  [NodeBT(dataBT='F', leftBT=NodeBT(dataBT='E', leftBT=EmptyBT(), rightBT=EmptyBT()),  ↙
    ↗ rightBT=NodeBT(dataBT='G', leftBT=EmptyBT(), rightBT=EmptyBT())), NodeBT(dataBT='A',  ↙
    ↗ leftBT=EmptyBT(), rightBT=EmptyBT()), NodeBT(dataBT='C', leftBT=EmptyBT(), rightBT=EmptyBT())]
len(ts) =  3
getDataBT(ts[0]) =  F
ts =  [NodeBT(dataBT='A', leftBT=EmptyBT(), rightBT=EmptyBT()), NodeBT(dataBT='C', leftBT=EmptyBT(),  ↙
    ↗ rightBT=EmptyBT()), NodeBT(dataBT='E', leftBT=EmptyBT(), rightBT=EmptyBT()),  ↙
    ↗ NodeBT(dataBT='G', leftBT=EmptyBT(), rightBT=EmptyBT())]
len(ts) =  4
getDataBT(ts[0]) =  A
ts =  [NodeBT(dataBT='C', leftBT=EmptyBT(), rightBT=EmptyBT()), NodeBT(dataBT='E', leftBT=EmptyBT(),  ↙
    ↗ rightBT=EmptyBT()), NodeBT(dataBT='G', leftBT=EmptyBT(), rightBT=EmptyBT()), EmptyBT(),  ↙
    ↗ EmptyBT()]
len(ts) =  5
getDataBT(ts[0]) =  C
ts =  [NodeBT(dataBT='E', leftBT=EmptyBT(), rightBT=EmptyBT()), NodeBT(dataBT='G', leftBT=EmptyBT(),  ↙
    ↗ rightBT=EmptyBT()), EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT()]
len(ts) =  6
getDataBT(ts[0]) =  E
ts =  [NodeBT(dataBT='G', leftBT=EmptyBT(), rightBT=EmptyBT()), EmptyBT(), EmptyBT(), EmptyBT(),  ↙
    ↗ EmptyBT(), EmptyBT(), EmptyBT()]
len(ts) =  7
getDataBT(ts[0]) =  G
ts =  [EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT()]
ts[0] is empty len(ts) =  8
ts =  [EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT()]
ts[0] is empty len(ts) =  7
ts =  [EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT()]
ts[0] is empty len(ts) =  6
ts =  [EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT()]
ts[0] is empty len(ts) =  5
ts =  [EmptyBT(), EmptyBT(), EmptyBT(), EmptyBT()]
ts[0] is empty len(ts) =  4
ts =  [EmptyBT(), EmptyBT(), EmptyBT()]
ts[0] is empty len(ts) =  3
ts =  [EmptyBT(), EmptyBT()]
ts[0] is empty len(ts) =  2
ts =  [EmptyBT()]
ts[0] is empty len(ts) =  1
ts =  []
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
  File  ↙
    ↗ "/Users/molyneux/MyData/Documents/OU/Courses/Computing/M269/M269TutorialResources/M269TutorialResources2022/
    ↗ line 548, in breadthBT04A
    if not (isEmptyBT(ts[0])) :
IndexError: list index out of range
Python3>>>
```

3.6.5 Breadth First V05

- There is one disadvantage of version 01
- The program traverses the entire left subtree before traversing the right subtree
- Bad news for large trees and very bad for infinite trees
- This version produces the traversal level by level

```
def labelsAtDepth(d : int, t : Tree) -> [T] :
  if isEmptyBT(t) :
    return []
  else :
    x = getDataBT(t)
    left = getLeftBT(t)
    right = getRightBT(t)
    if d == 0 :
      return [x]
    else :
      return (labelsAtDepth((d-1),left) + labelsAtDepth((d-1),right))
```

- Breadth first traversal with `labelsAtDepth`
- Version based on Sannella et al. (2022, page 261) *Introduction to Computation: Haskell, Logic and Automata*

```
def bfTraverseByLevels(t : Tree) -> [T] :
  return bfTbyL(0,t)

def bfTbyL(d : int, t : Tree) -> [T] :
  xs = labelsAtDepth(d,t)
  if xs == [] :
    return []
  else :
    return (xs + bfTbyL((d+1),t))
```

ToC

3.7 Recursive Thinking

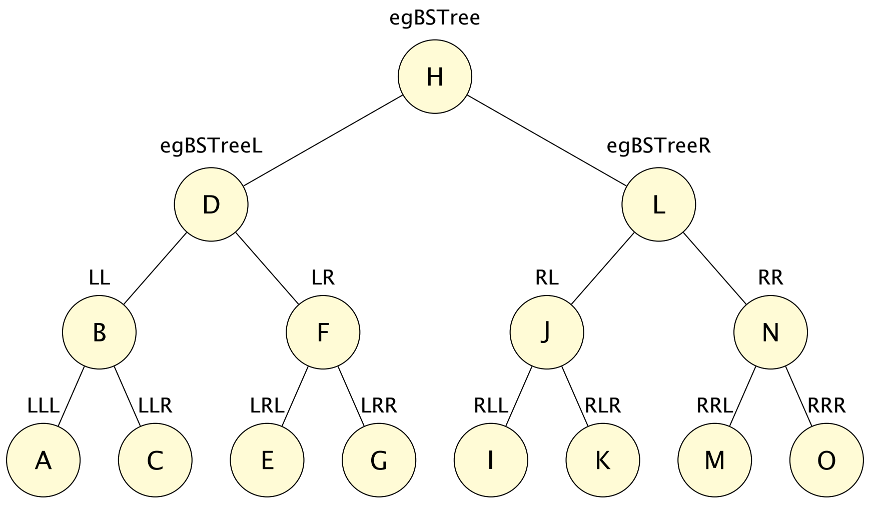
- Since Binary Trees are defined recursively as either an empty tree or a node with an item and two sub-trees it is often easier to define a recursive program for a function on Binary Trees
- This (and other sections) give examples of recursive thinking with binary trees
- In some cases we give an iterative version as well and demonstrate deriving an iterative version from the recursive version
- Often the iterative version is more complex than the recursive version

3.7.1 Lowest Common Ancestor

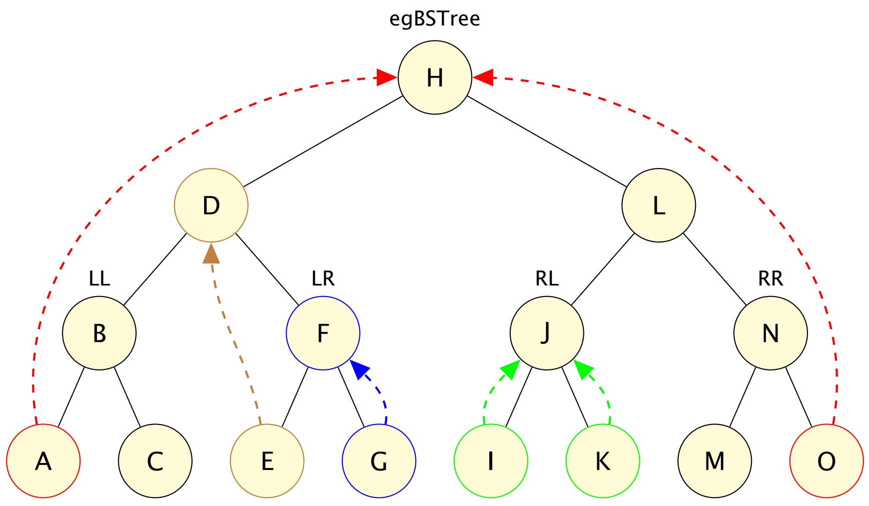
- The **Lowest Common Ancestor** of two nodes k1 and k2 in a tree T is the lowest node that is an ancestor of both k1 and k2
- This is a famous problem — see Wikipedia: Lowest common ancestor
- These notes outline a recursive approach to implementing an algorithm
- Note that there are iterative approaches but they tend to be more complex or assume each node already has a pointer to its parent — you could write a program to add pointers to parents as a separate exercise

- There are some subtle points about the *Lowest Common Ancestor* problem that catch students out.
- The two nodes have to be in the tree at the first call to the algorithm — otherwise you can get some incorrect answers
- A node has to be allowed to be regarded as an ancestor (or descendent) to itself otherwise the algorithm goes wrong
- The student has to include code for checking that a given node or key is in the tree (which could of itself be a question part)

Example egBSTree



- **Questions**
- What should the `lowest_common_ancestor` applied to `egBSTree` and the following arguments, return ?
- (egBSTree, ’A’, ’O’), (egBSTree, ’I’, ’K’)
- (egBSTree, ’F’, ’G’), (egBSTree, ’E’, ’D’)
- (egBSTree, ’A’, ’X’), (egBSTree, ’X’, ’Y’)



- `reportLCA(egBSTree, 'A', 'O')` different branches
- `reportLCA(egBSTree, 'I', 'K')` different branches

- `reportLCA(egBSTree, 'F', 'G')` same branch
- `reportLCA(egBSTree, 'E', 'D')` same branch
- This example uses the ADT given below — this is different to the M269 representation but equivalent
- This emphasises that a Binary Tree is a union of an Empty Tree and the set of all non-empty nodes, with an item and two subtrees
- It also has the advantage of a default print representation that is useful
- The code is in file `M269BinaryTrees2025LCA.py`

```python
from collections import namedtuple

EmptyBT = namedtuple('EmptyBT',[])

NodeBT = namedtuple('NodeBT'
                   ,['dataBT','leftBT','rightBT'])

# Tree type

from typing import TypeVar,Union,NewType

T = TypeVar('T')
Tree = NewType('Tree',Union[EmptyBT,NodeBT])
```

- Here are the Binary Tree operations used in these notes

```python
def mkEmptyBT() -> Tree :
  return EmptyBT()

def mkNodeBT(x : T,t1 : Tree,t2 : Tree) -> Tree :
  return NodeBT(x,t1,t2)

def isEmptyBT(t : Tree) -> bool:
  return t == EmptyBT()

def getDataBT(t : Tree) -> T:
  if isEmptyBT(t):
    raise RuntimeError("getDataBT applied to EmptyBT()")
  else:
    return t.dataBT

def getLeftBT(t : Tree) -> Tree :
  if isEmptyBT(t):
    raise RuntimeError("getLeftBT applied to EmptyBT()")
  else:
    return t.leftBT

def getRightBT(t : Tree) -> Tree :
  if isEmptyBT(t):
    raise RuntimeError("getRightBT applied to EmptyBT()")
  else:
    return t.rightBT
```

- If `t` is empty then return an empty tree (representing fail)
- If one of the two input keys equals the key at `t` then return `t`
- Otherwise recurse with two further calls to `lca(getLeftBT(t), k1, k2)` and `lca(getRightB` `k1, k2)`

  Either the two nodes are in separate branches or the same branch
- Note that the following code cannot be used directly in any M269 exercise (since it uses different notation)

```
136 def lca(t  Tree, k1: T, k2: T) -> Tree :
137   if isEmptyBT(t) :
138     return mkEmptyBT()
139   elif (getDataBT(t) == k1 or getDataBT(t) == k2) :
140     return t
141   else :
142     leftLCA  = lca(getLeftBT(t),k1,k2)
143     rightLCA = lca(getRightBT(t),k1,k2)
144     if (not (isEmptyBT(leftLCA))
145         and not (isEmptyBT(rightLCA))) :
146       return t
147     else :
148       return (leftLCA if not (isEmptyBT(leftLCA))
149                       else rightLCA)
```

- Note this code cannot be used directly in any M269 exercise since it will be rejected
- We initially have to check both nodes are in the tree and print some sensible string as a result

```
151 def isInTree(t: Tree, k: T) -> bool :
152   if isEmptyBT(t) :
153     return False
154   elif (getDataBT(t) == k) :
155     return True
156   else :
157     return (isInTree(getLeftBT(t), k)
158             or isInTree(getRightBT(t), k))


161 def reportLCA(t: Tree, k1: T, k2: T) -> str :
162   if (not isInTree(t, k1) or not isInTree(t, k2)) :
163     return ("Not␣both␣k1␣\'" + k1 + "\'␣and␣k2␣\'"
164             + k2 + "\'␣are␣in␣the␣tree")
165   else :
166     valueLCA = lca(t, k1, k2)
167     if isEmptyBT(valueLCA) :
168       return str(valueLCA)
169     else :
170       reportStr = ("Key␣value␣of␣LCA␣is␣\'"
171                    + str(getDataBT(valueLCA)) + "\'")
172       return reportStr
```

- We should now work through the above examples to see what happens
- We should also indicate what happens if we run `lca` with one or both nodes not in the initial tree
- **LCA References**
- Wikipedia: Lowest common ancestor
- Lowest Common Ancestor in a Binary Tree
- Wikipedia: Schieber-Vishkin algorithm
- **Evaluate** showing which line in the code was used at each stage (or *reduction* step)
- `lca(egBSTree, 'A', 'O')`
- `lca(egBSTreeL, 'D', 'E')`
- `lca(egBSTreeL, 'X', 'Y')`
- `lca(egBSTree, 'A', 'X')`

```
 lca(egBSTree, 'A', 'O')
 = lca(egBSTreeL,'A','O')                # by line 141
   lca(egBSTreeR,'A','O')
```

```
lca(egBSTreeL,'A','O')
= lca(egBSTreeLL,'A','O')                # by line 141
  lca(egBSTreeLR,'A','O')

lca(egBSTreeLL,'A','O')
= lca(egBSTreeLLL,'A','O')               # by line 141
  lca(egBSTreeLLR,'A','O')

lca(egBSTreeLLL,'A','O')
= egBSTreeLLL                            # by line 139
```

```
lca(egBSTreeLR,'A','O')
  = mkEmptyBT()                          # by lines 141*3, 147*2
```

```
lca(egBSTreeR,'A','O')
= egBSTreeRRR                            # by similar reasoning
```

```
lca(egBSTree, 'A', 'O')
= egBSTree                               # by line 144
```

```
Python3>>> from M269BinaryTrees2025LCA import *
Python3>>> reportLCA(egBSTree, 'A','O')
"Key␣value␣of␣LCA␣is␣'H'"
Python3>>> reportLCA(egBSTree, 'D','E')
"Key␣value␣of␣LCA␣is␣'D'"
Python3>>> reportLCA(egBSTree, 'I','K')
"Key␣value␣of␣LCA␣is␣'J'"
Python3>>> reportLCA(egBSTree, 'F','G')
"Key␣value␣of␣LCA␣is␣'F'"
Python3>>> lca(egBSTree, 'Y','X')
EmptyBT()
Python3>>> lca(egBSTree, 'A','X')
NodeBT(dataBT='A', leftBT=EmptyBT(), rightBT=EmptyBT())
Python3>>> reportLCA(egBSTree,'Y','X')
"Not␣both␣k1␣'Y'␣and␣k2␣'X'␣are␣in␣the␣tree"
Python3>>> reportLCA(egBSTree,'A','X')
"Not␣both␣k1␣'A'␣and␣k2␣'X'␣are␣in␣the␣tree"
Python3>>>
```

- The above is why we need `isInTree(t,k)` and `reportLCA(t,k1,k2)`

ToC

ToC

ToC

4 Iterative Tree Traversals

- We have used recursion in our implementation of algorithms on binary trees
- This has made it easier to produce correct and fairly simple implementations
- This is mainly because the binary tree data structure is itself defined recursively
- A binary tree is either an empty tree or a node with a data item and two subtrees.
- However the efficiency of this approach will depend on how the chosen programming language is implemented.
- We are using Python and, while Python permits recursion, it does not do some of the optimisations available in other languages, especially pure functional languages (such as Haskell).
- Hence you may find some Python texts down play the use of recursion.

- It is always possible to convert a recursive program into one that just uses iteration with `while` loops or (possibly) `for` loops
- We give below examples of the depth first tree traversals translated from their recursive forms to non-recursive.
- Note that this subsection is for illustration only and you would not be expected to be able to reproduce the code or convert other recursive code.

4.1 Iterative InOrder Traversal

- Here is the original recursive version (from line 311 on page 23)

```
311 def inOrderBT(t) :
312   if isEmptyBT(t) :
313     return []
314   else :
315     return (inOrderBT(getLeftBT(t)) + [getDataBT(t)]
316             + inOrderBT(getRightBT(t)))
```

- We start with the recursive version but with an accumulating result.

```
335 def inOrderBT0(t) :
336   result = []
337   if not isEmptyBT(t) :
338     result = result + (inOrderBT0(getLeftBT(t)))
339     result.append(getDataBT(t))
340     result = result + (inOrderBT0(getRightBT(t)))
341   return result
```

- Turn the final (*almost* tail recursive) call into a `while` loop

```
343 def inOrderIterBT1(t) :
344   result = []
345   while not isEmptyBT(t) :
346     result = result + (inOrderIterBT1(getLeftBT(t)))
347     result.append(getDataBT(t))
348     t = getRightBT(t)
349   return result
```

- The term *almost* since the last operation is the addition `(+)` but that could be wrapped into the last call
- There is now one recursive call.
- Create a stack to store the function call context.
- In the loop have a conditional to determine whether to store a new context and make the left sub tree the current node or if we are returning, with the appropriate code.

```
351 def inOrderIterBT2(t) :
352   result = []
353   stack = []
354   while (not (stack == []) or not isEmptyBT(t)) :
355     if not isEmptyBT(t) :
356       stack.append(t)
357       t = getLeftBT(t)
358     else:
359       t = stack.pop()
360       result.append(getDataBT(t))
361       t = getRightBT(t)
362   return result
```

Algorithm Description

- inOrderIterBT2 takes a tree t and returns a list of items at the nodes, depth first from left to right
  1. Initialise result to an empty list, and stack, for the stack of trees to visit, to an empty list
  2. While stack is not empty or t is not the empty tree
     (a) If t is not empty, append t to stack and assign t to be its left sub tree — this is the left recursion
     (b) Otherwise make t the top of the stack (and remove it), append the item at its node to result and make t to be its right sub tree
  3. Finally return result

ToC

4.2 Iterative PreOrder Traversal

- Here is the original recursive version (from line 318 on page 23)

```
318 def preOrderBT(t) :
319   if isEmptyBT(t) :
320     return []
321   else :
322     return ([getDataBT(t)] + preOrderBT(getLeftBT(t))
323             + preOrderBT(getRightBT(t)))
```

- Start with recursive version with accumulating result.

```
364 def preOrderBT0(t) :
365   result = []
366   if not isEmptyBT(t) :
367     result.append(getDataBT(t))
368     result = result + (preOrderBT0(getLeftBT(t)))
369     result = result + (preOrderBT0(getRightBT(t)))
370   return result
```

- Turn the final (*almost* tail recursive) call into a while loop.

```
372 def preOrderIterBT1(t) :
373   result = []
374   while not isEmptyBT(t) :
375     result.append(getDataBT(t))
376     result = result + (preOrderIterBT1(getLeftBT(t)))
377     t = getRightBT(t)
378   return result
```

- There is now one recursive call.
- Create a stack to store the function call context.
- In the loop have a conditional to determine whether to store a new context and make the left sub tree the current node or if we are returning, with the appropriate code

```
380 def preOrderIterBT2(t) :
381   result = []
382   stack = []
383   while (not (stack == []) or not isEmptyBT(t)) :
384     if not isEmptyBT(t) :
385       result.append(getDataBT(t))
386       stack.append(t)
387       t = getLeftBT(t)
388     else :
389       t = stack.pop()
390       t = getRightBT(t)
```

```
391     return result
```

ToC

4.3 Iterative PostOrder Traversal

- Here is the original recursive version (from line 325 on page 23)

```
325 def postOrderBT(t):
326   if isEmptyBT(t):
327     return []
328   else:
329     return (postOrderBT(getLeftBT(t))
330             + postOrderBT(getRightBT(t)) + [getDataBT(t)])
```

- Start with recursive version with accumulating result.

```
393 def postOrderBT0(t) :
394   result = []
395   if not isEmptyBT(t) :
396     result = result + (postOrderIterBT1(getLeftBT(t)))
397     result = result + (postOrderIterBT1(getRightBT(t)))
398     result.append(getDataBT(t))
399   return result
```

- There is now no final (tail recursive) call. We have to mimic the call stack *carefully*

```
401 def postOrderIterBT1(t) :
402   result = []
403   if isEmptyBT(t) :
404     return result
405   stack = []
406   while not (stack == []) or (not isEmptyBT(t)) :
407     while not isEmptyBT(t) :
408       if not isEmptyBT(getRightBT(t)) :
409         stack.append(getRightBT(t))
410       stack.append(t)
411       t = getLeftBT(t)
412     t = stack.pop()
413     if ((not isEmptyBT(getRightBT(t)))
414         and (stack != [] and stack[-1] is getRightBT(t))) :
415       tr = stack.pop()
416       stack.append(t)
417       t = getRightBT(t)
418     else :
419       result.append(getDataBT(t))
420       t = mkEmptyBT()  # To avoid infinite loop - it is t.rightBT
421   return result
```

Algorithm Description

1. Initialise `result` to an empty list.
2. If `t` is empty then return `result` (not really needed since the loop would take care of this)
3. Initialise `stack`, for the stack of trees to visit, to an empty list
4. While `stack` is not empty or `t` is not the empty tree
   (a) While `t` is not the empty tree
      - If the right sub tree of `t` is not empty, push it on to `stack`
      - Append `t` to `stack`
      - Assign `t` its left sub tree

(b) Pop a node from `stack` and set it as `t`

(c) If the popped node has a non empty right child and the right child is at the top of `stack`

- Remove the right child from the `stack`
- Push the current node `t` on to `stack`
- Set `t` to be `t`'s right child

(d) Otherwise

- Append the data at the root of `t` to `result`
- Set `t` to `Empty()` — marking `t` as visited, prevents infinite looping (it is `t.rightBT`)

5. Finally return `result`

Concluding Points

- Recursive versions are easier to get right.
- Iterative versions mimic the stack of recursive function calls.
- Other non-recursive versions use different data structures with pointers to parent nodes. The code is still more complex (and error prone) compared to the recursive versions.

ToC

Commentary 3

3 Binary Search Trees

- Binary trees with the *binary search tree* property
- Inserting a node
- Other BST operations
- Deletion — investigating choices

ToC

5 Binary Search Trees

5.1 Binary Search Trees — Definition

- A binary search tree (BST) is a binary tree with the *binary search tree* property:
  1. The left sub tree contains nodes with keys less than the root node
  2. The right sub tree contains nodes with keys greater than the root node
  3. The left and right sub trees must also be binary search trees
  4. No nodes with duplicate keys

5. An empty tree is a binary search tree
6. Nothing else is a binary search tree

- The data at each node is to contain a key and any values. The operations required for a BST will include:

  `insertBST()`, `inBST()`, `isBSTree()`, `insertListBST()`, `buildBST()`, `deleteBST()`

Binary Search Trees — Motivation

- A *perfect* binary tree of height h will have $2^h - 1$ nodes
- This means that there will be at most $\log_2(n+1)$ steps from the root of the tree to any node in the tree.
- This provides the basis for efficient searching if we give an appropriate structure to the tree — a *Binary Search Tree (BST)*
- However we have to keep the BST as near to a perfect tree otherwise we lose the advantage

Activity 6 Nodes of Perfect Tree

- Justify the statement that a *perfect* binary tree of height h will have $2^h - 1$ nodes

Go to Answer

Answer 6 Nodes of Perfect Tree

- There are many ways of showing that a *perfect* binary tree of height h will have $2^h - 1$ nodes — here is one way
- Let N_h be the number of nodes in a perfect tree and L_h be the number of leaves in the same tree.
- Then we have $L_0 = 0$, $L_1 = 1$, $L_2 = 2$, $L_3 = 4, \ldots$ and in general $L_h = 2^{h-1}$, $h \geqslant 1$
- Now $N_h = L_1 + L_2 + \cdots + L_h = 2^0 + 2^1 + \cdots + 2^{h-1}$
- $2N_h = 2^1 + 2^2 + \cdots + 2^{h-1} + 2^h$
- Subtract the N_h from $2N_h$ and we get $N_h = 2^h - 1$
- Notice that $N_h = 1 + 2 \times N_{h-1}$ — when we consider the performance we will use similar recurrence relations

Go to Activity

ToC

5.2 Inserting a Node

Inserting a Node — Description

- The function that takes a item (key and payload) and an existing BST has to return a new binary tree with the item inserted and the new tree must be a BST. We deal with each possible binary tree: an empty tree and a non-empty tree:
- To insert an item into an empty tree, we return a new tree which is a singleton node with the item and two empty sub-trees.

- To insert an item, with key *k*, into a tree which is a node with an item with key *p* and two sub-trees then we have two possibilities
  - If *k* is less than *p* then insert the item in the left sub-tree
  - If *k* is greater than *p* then insert the item in the right sub-tree
- We are going to assume duplicate keys are not allowed — in practice, we might have an error message or update the payload of an item with the same key but we are keeping things simple here.

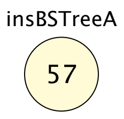
Binary Search Tree — Inserting a Node

```python
def insertBST(x,t):
  if isEmptyBT(t):
    return mkNodeBT(x,mkEmptyBT(),mkEmptyBT())
  else:
    y = getDataBT(t)
    if x < y:
      return mkNodeBT(y,insertBST(x,getLeftBT(t)),getRightBT(t))
    elif x > y:
      return mkNodeBT(y,getLeftBT(t),insertBST(x,getRightBT(t)))
    else:
      return t
```

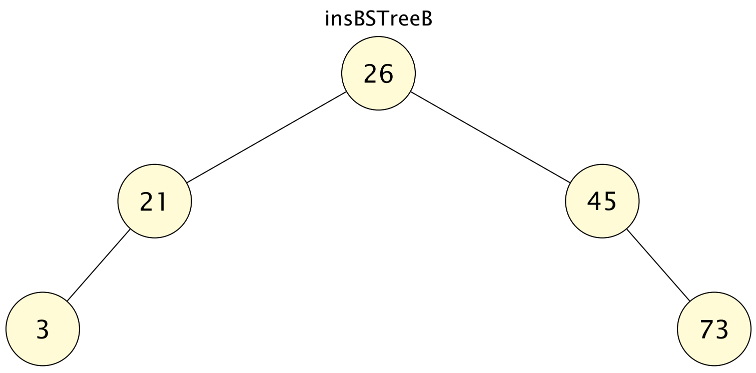
Activity 7 Inserting an Item

- Draw diagrams of the binary search trees that result from inserting an item with key 28 into each of the three BSTs in the diagrams below of insBSTreeA, insBSTreeB, insBSTreeC

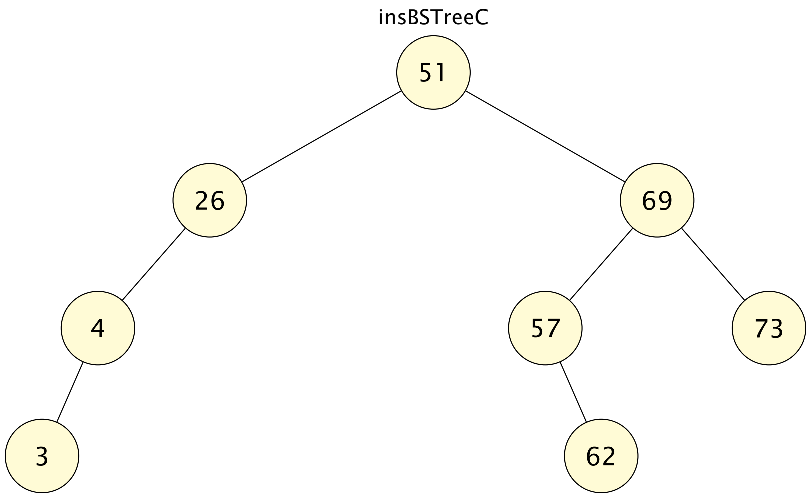
insBSTreeA



insBSTreeB



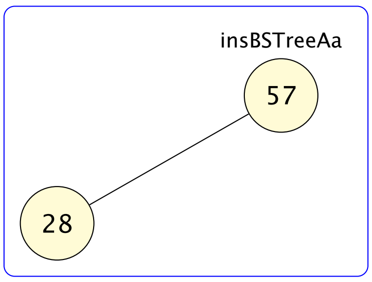
insBSTreeC



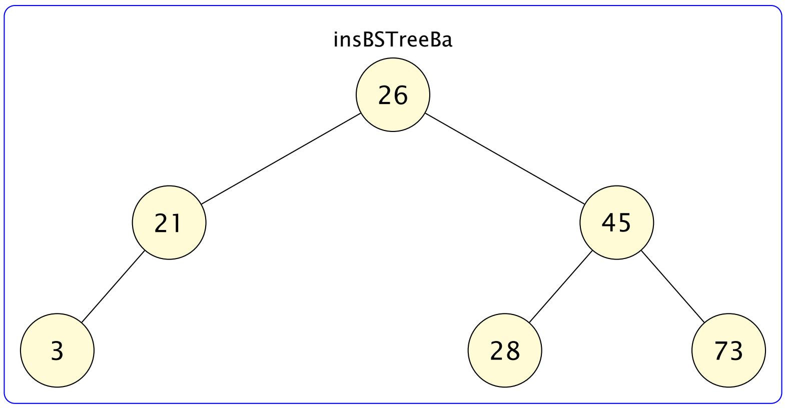
Go to Answer

Answer 7 Inserting an Item

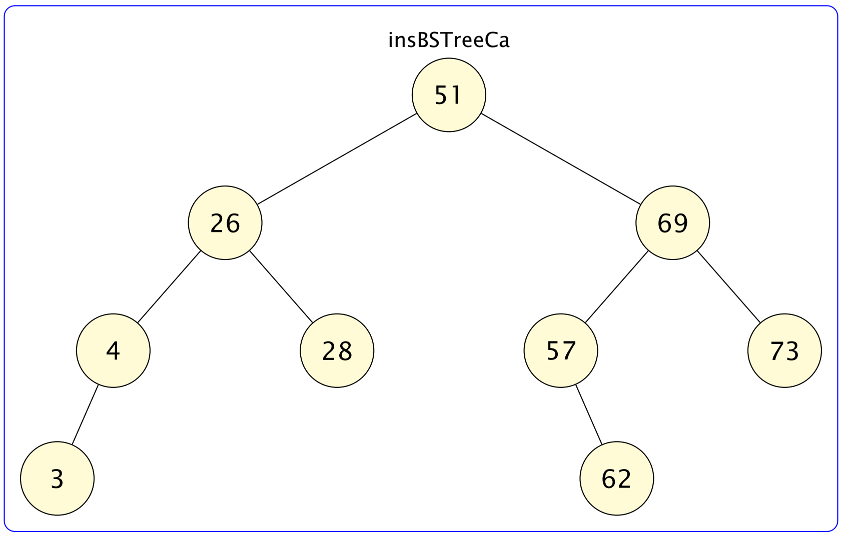
insBSTreeAa



insBSTreeBa



insBSTreeCa



Go to Activity

5.3 BST Operations

Activity 8 Membership

- Write Python code for a function, `inBST(k,t)` which take a key, *k*, and a BST, *t* and returns `True` if an item with key *k* is in the tree and `False` otherwise

Go to Answer

Answer 8 Membership

```
def inBST(k,t):
  if isEmptyBT():
    return False
  else:
    p = getDataBT(t)
    if k < p:
      return inBST(k,getLeftBT(t))
    elif k > p:
      return inBST(k,getRightBT(t))
    else:
      return True
```

Go to Activity

5.3.1 Testing a BST

Testing if a Binary Tree is a BST

- One strategy for this might be to do an in-order traversal of the tree and check that the list returned was an ordered list.
- The ordering relation is `(<)` for strict ordering and no duplicates

```
def isBSTree(t):
  return orderedList(inOrderBT(t))

def orderedList(xs):
  return (len(xs) <= 1
         or (xs[0] < xs[1] and orderedList(xs[1:])))
```

5.3.2 List to BST

Building a Binary Search Tree from a list of items

- We *could* insert a list of items one by one from the list in turn — here is the Python code:

```python
def insertListBST(xs,t) :
  if xs == [] :
    return t
  else :
    return insertListBST (xs[1:], insertBST(xs[0],t))
```

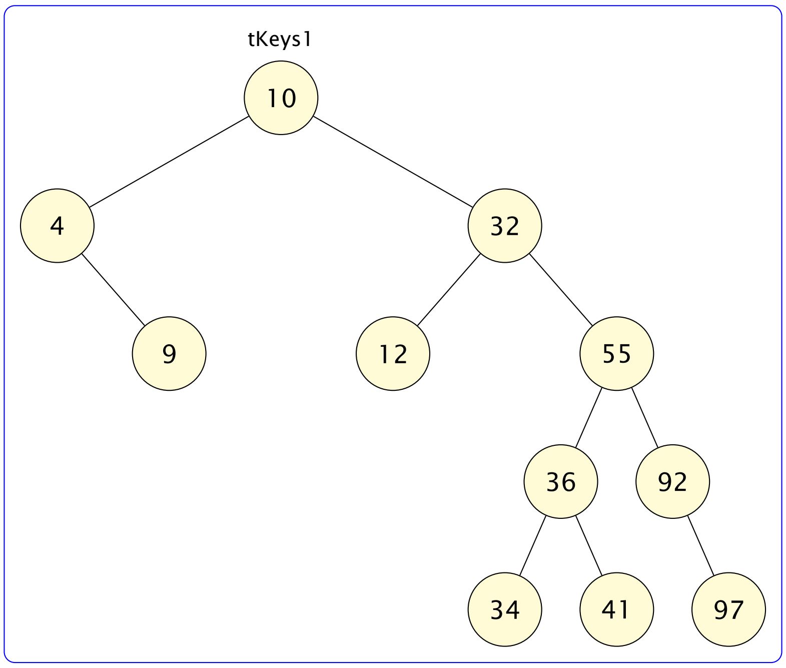
- However, see what happens when we insert various lists — how *compact* is the resulting tree ?

Activity 9 Insert List

- For the following lists of keys, draw the diagrams of the trees produced when `insertListBST()` is used to produce the trees from the lists inserting the keys into an initially empty tree
- `keys1 = [10, 4, 32, 12, 9, 55, 92, 97, 36, 41, 34]`
- `keys2 = [4, 9, 10, 12, 32, 97, 92, 55, 41, 34, 32]`
- `keys3 = [34, 10, 9, 4, 32, 12, 55, 41, 36, 97, 92]`

Go to Answer

Answer 9 Insert List `tKeys1 = insertListBST(keys1, mkEmptyBT())`

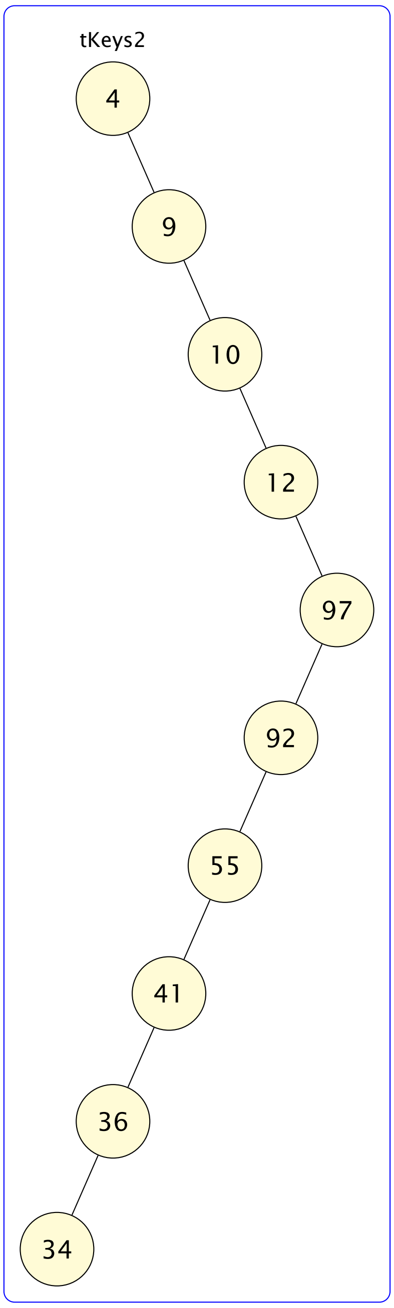


```python
tKeys1 = mkNodeBT(10,
           mkNodeBT(4,
             mkEmptyBT(),
             mkNodeBT(9, mkEmptyBT(), mkEmptyBT())),
           mkNodeBT(32,
             mkNodeBT(12, mkEmptyBT(), mkEmptyBT()),
             mkNodeBT(55,
               mkNodeBT(36,
                 mkNodeBT(34, mkEmptyBT(), mkEmptyBT()),
                 mkNodeBT(41, mkEmptyBT(), mkEmptyBT())),
               mkNodeBT(92,
                 mkEmptyBT(),
                 mkNodeBT(97, mkEmptyBT(), mkEmptyBT())))))
```

```
tKeys1Test = (tKeys1
              == insertListBST(keys1, mkEmptyBT()))
```

- Note that when the Python interpreter prints tKeys1 it includes field names and values and has no line breaks.

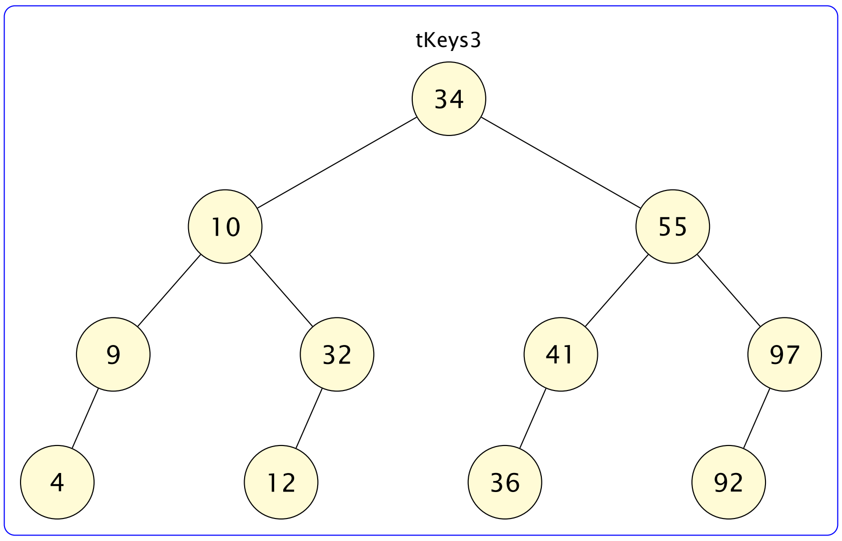
Answer 9 Insert List tKeys2 = insertListBST(keys2, mkEmptyBT())



```
  tKeys2 = mkNodeBT(4, mkEmptyBT(),
             mkNodeBT(9, mkEmptyBT(),
               mkNodeBT(10, mkEmptyBT(),
                 mkNodeBT(12, mkEmptyBT(),
                   mkNodeBT(32, mkEmptyBT(),
                     mkNodeBT(97,
                       mkNodeBT(92,
                         mkNodeBT(55,
                           mkNodeBT(41,
                             mkNodeBT(36,
                               mkNodeBT(34, mkEmptyBT(),
                                            mkEmptyBT()),
                               mkEmptyBT()),
                             mkEmptyBT()),
                           mkEmptyBT()),
                         mkEmptyBT()),
                       mkEmptyBT())))))))

  tKeys2Test = (tKeys2
                == insertListBST(keys2, mkEmptyBT()))
```

Answer 9 Insert List `tKeys3 = insertListBST(keys3, mkEmptyBT())`



```
tKeys3 = mkNodeBT(34,
           mkNodeBT(10,
             mkNodeBT(9,
               mkNodeBT(4, mkEmptyBT(), mkEmptyBT()),
               mkEmptyBT()),
             mkNodeBT(32,
               mkNodeBT(12, mkEmptyBT(), mkEmptyBT()),
               mkEmptyBT())),
           mkNodeBT(55,
             mkNodeBT(41,
               mkNodeBT(36, mkEmptyBT(), mkEmptyBT()),
               mkEmptyBT()),
             mkNodeBT(97,
               mkNodeBT(92, mkEmptyBT(), mkEmptyBT()),
               mkEmptyBT())))

tKeys3Test = (tKeys3
              == insertListBST(keys3, mkEmptyBT()))
```

- Notice that tree tKeys2 has height equal to the number of items 11
- The structure might as well be a list
- In this case the tree structure would not be more efficient than a list for searching.
- The height of tKeys3 is 4 which is as compact a tree with the number of items between 8 and 15.
- tKeys2 shows inserting a list in even partly sorted order results in the worst case for efficiency.
- If a binary search tree is built from insertion of a list of random data then it can be shown that the expected height of the tree is $O(\log n)$
- The proof of this requires knowledge of statistics outside the remit of this course — if interested, a proof is in Cormen et al. (2009, page 300) *Theorem 12.4* and Cormen et al. (2022, page 328) *Problem 12-3*

Go to Activity

Building a Compact BST

- To produce as compact a tree as possible, we could we could do the following:
- Sort the list
- Find the middle item in the sorted list
- Construct a binary tree node with the middle item as the data
- The left and right sub-trees should be formed by recursively building binary trees from the front and back parts of the sorted list
- Below is an implementation in Python

```python
def buildBST(xs) :
  return bBST(mkEmptyBT(), sorted(xs))


def bBST(t,xs) :
  if xs == [] :
    return t
  else :
    half = len(xs) // 2
    x = xs[half]
    frontxs = xs[:half]
    backxs = xs[half+1:]
    if isEmptyBT(t) :
      return (mkNodeBT(x,
                bBST(mkEmptyBT(),frontxs),
                bBST(mkEmptyBT(),backxs)))
    else :
      errMsg = ("bBST:␣Trying␣to␣insert" + str(xs)
             + "␣into␣nonempty␣tree" + str(t))
      raise RuntimeError(errMsg)
```

```
Python3>>> xs = [1, 9, 2, 8, 3, 7, 4, 6, 5]
Python3>>> buildBST(xs)
  NodeBT(dataBT=5,
    leftBT=NodeBT(dataBT=3,
             leftBT=NodeBT(dataBT=2,
                      leftBT=NodeBT(dataBT=1,
                               leftBT=EmptyBT(),
                               rightBT=EmptyBT()),
                      rightBT=EmptyBT()),
              rightBT=NodeBT(dataBT=4,
                        leftBT=EmptyBT(),
                        rightBT=EmptyBT())),
    rightBT=NodeBT(dataBT=8,
              leftBT=NodeBT(dataBT=7,
                       leftBT=NodeBT(dataBT=6,
                                leftBT=EmptyBT(),
                                rightBT=EmptyBT()),
                       rightBT=EmptyBT()),
              rightBT=NodeBT(dataBT=9,
                        leftBT=EmptyBT(),
                        rightBT=EmptyBT())))
Python3>>>
```

- Note that when the Python interpreter prints a namedtuple it includes field names and values and has no line breaks

ToC

5.4 BST Deleting a Node

- Deleting an item from a binary search tree involves more choices than insertion
- *Initial insight*

  - Find the node with the item (key) by following left or right sub trees
  - Delete the item by joining the two sub trees of the node
  - If the item is not in the tree, just return mkEmptyBT()

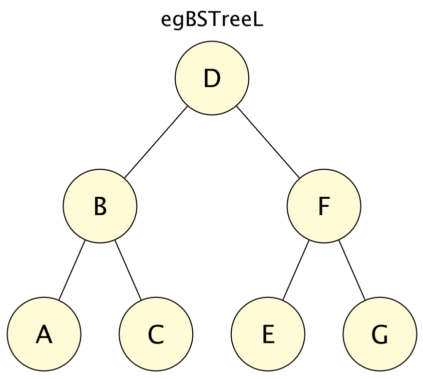
- The tricky bit is deciding how to join the two sub trees while keeping the binary search tree property and keeping the tree compact (otherwise we lose the advantage of a binary search tree).
- The following presents three alternatives which each use some information about binary search trees — each version is correct but the later versions produce a more compact tree.
- Below is the initial insight implemented in Python:

Deleting an Item from a BST — Python

```
def deleteBST(x, t):
  if isEmptyBT(t):
    return mkEmptyBT()
  else:
    y = getDataBT(t)
    leftT = getLeftBT(t)
    rightT = getRightBT(t)
    if x < y:
      return mkNodeBT(y, (deleteBST(x,leftT)), rightT)
    elif x > y:
      return mkNodeBT(y, leftT, (deleteBST(x,rightT)))
    else:
      return joinBST(leftT, rightT)
```

Example Binary Search Tree Deletion

- We now investigate different ways of joining two subtrees with the function joinBST(leftT, rightT)
- We shall use the small tree egBSTreeL to illustrate the choices deleting the node with key D



- Here is a Python implementation of the tree egBSTreeL

```
egBSTreeL = mkNodeBT('D',
              mkNodeBT('B',
                mkNodeBT('A',mkEmptyBT(),mkEmptyBT()),
                mkNodeBT('C',mkEmptyBT(),mkEmptyBT())
              ),
              mkNodeBT('F',
                mkNodeBT('E',mkEmptyBT(),mkEmptyBT()),
                mkNodeBT('G',mkEmptyBT(),mkEmptyBT())
              ))
```

joinBST Version 1

- joinBST1 lists out all the elements of the left sub tree and inserts them in the right subtree.
- It does an in-order traversal of the left sub tree and then inserts the resulting list of items in the right subtree.

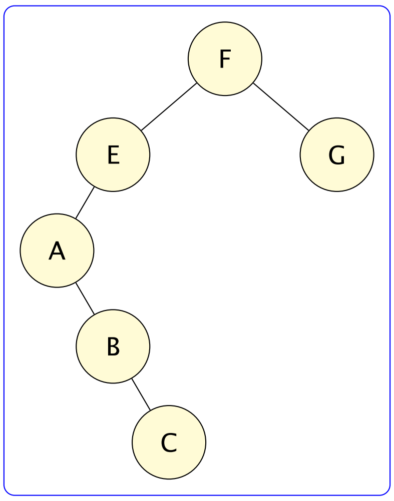
```python
def joinBST1(leftT, rightT):
  if isEmptyBT(leftT):
    return rightT
  else:
    return (insertListBST(inOrderBT(leftT), rightT))
```

Activity 10 joinBST Version 1

- Draw the diagram of the tree resulting from deleting D with joinBST1
- Why is joinBST1 not a good strategy ?

Go to Answer

Answer 10 joinBST Version 1



```python
delBSTreeJoin1 = joinBST1(egBSTreeLL,egBSTreeLR)

delBSTreeJoin1ans = \
  NodeBT(dataBT='F',
    leftBT=NodeBT(dataBT='E',
             leftBT=NodeBT(dataBT='A',
                      leftBT=EmptyBT(),
                      rightBT=NodeBT(dataBT='B',
                                leftBT=EmptyBT(),
                                rightBT=NodeBT(dataBT='C',
                                          leftBT=EmptyBT(),
                                          rightBT=EmptyBT()))),
              rightBT=EmptyBT()),
    rightBT=NodeBT(dataBT='G',
              leftBT=EmptyBT(),
              rightBT=EmptyBT()))

delBSTreeJoin1test = delBSTreeJoin1 == delBSTreeJoin1ans
```

Go to Activity

joinBST Version 2

- joinBST1 results in a near linear structure and is not as compact as it could be.
- The first definition made no use of our knowledge of binary search trees.
- We know that:

```
maxKey leftT < minKey rightT
```

since they were subtrees of the original Binary Search tree, egBSTreeL

- In particular we therefore know that the root of the left subtree is less than any item in the right subtree.
- So we attach the left subtree under the smallest element in the right sub tree.

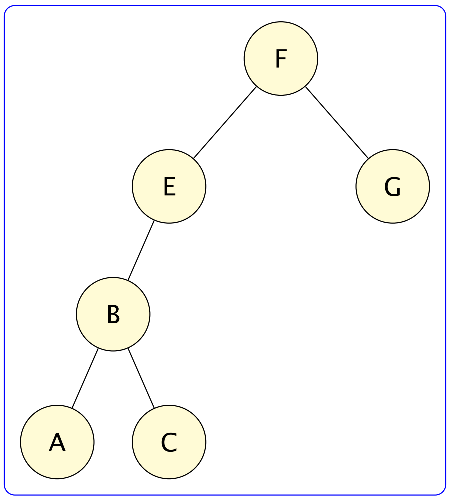
```
672 def joinBST2(leftT, rightT):
673   if isEmptyBT(rightT):
674     return leftT
675   else:
676     return (mkNodeBT(getDataBT(rightT),
677                      joinBST2(leftT, getLeftBT(rightT)),
678                      getRightBT(rightT)))
```

Activity 11 joinBST Version 2

- Draw the diagram of the tree resulting fron deleting D with joinBST2
- Can you see how we can improve on joinBST2 ?

Go to Answer

Answer 11 joinBST Version 2



```
681 delBSTreeJoin2 = joinBST2(egBSTreeLL,egBSTreeLR)

683 delBSTreeJoin2ans = NodeBT('F',
684                       NodeBT('E',
685                         NodeBT('B',
686                           NodeBT('A', EmptyBT(), EmptyBT()),
687                           NodeBT('C', EmptyBT(), EmptyBT())),
688                         EmptyBT()),
689                       NodeBT('G', EmptyBT(), EmptyBT()))

691 delBSTreeJoin2test = (delBSTreeJoin2 == delBSTreeJoin2ans)
```

- To see how this works we shall do a step by step evaluation
- Follow the function code for joinBST2 from line 672 on page 50
- Note from the definitions of delBSTreeJoin1test (from line 668 on page 49) and delBSTreeJoin2test (from line 691 on page 50) we can use the field names or leave them out
- **Step 1** In the first call to joinBST2 the leftT is the tree rooted at B and the rightT is the tree rooted at F

- Line 673 tests if the second argument to joinBST2 is an empty tree
- Since it is not empty, joinBST2 evaluates to the value at line 676
- Hence we have

```
mkNodeBT('F',
         joinBST2(leftT, getLeftBT(rightT)),
         getRightBT(rightT))
```

- **Step 2** Since the return value has a recursive call to joinBST2 we need to evaluate that.
- Its second argument is rightT.leftBT which is the tree rooted at E
- Line 673 tests if the first argument to joinBST2 is an empty tree
- Since it is not empty, joinBST2 evaluates to the value at line 676
- Hence we have

```
mkNodeBT('F',
         mkNodeBT('E',
                  joinBST2(leftT, getLeftBT(getLeftBT(rightT))),
                  getRightBT(getLeftBT(rightT))),
         getRightBT(rightT))
```

- **Step 3** We have to evaluate a further recursive call
- The second argument to the recursive call to joinBST2 is rightT.leftBT.leftBT which is EmptyBT(), so the recursive call to joinBST2 evaluates to leftT
- rightT.leftBT.rightBT evaluates to EmptyBT()
- Hence we have

```
mkNodeBT('F',
         mkNodeBT('E',
                  leftT,
                  mkEmptyBT()),
         getRightBT(rightT))
```

- Hence the final value is

```
NodeBT('F',
  NodeBT('E',
    NodeBT('B',
      NodeBT('A', EmptyBT(), EmptyBT()),
      NodeBT('C', EmptyBT(), EmptyBT())),
    EmptyBT()),
  NodeBT('G', EmptyBT(), EmptyBT()))
```

- Doing a step by step evaluation of recursive function calls should help you get a better feel for recursive thinking.
- We can do better than this — see the following.

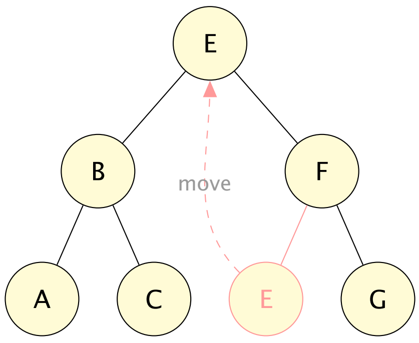
Go to Activity

joinBST Final Version

- We can make better use of our knowledge of Binary Search trees
- We know that:

```
maxKey leftT < root key < minKey rightT
```

- Hence we can promote the minimum item in the right subtree to be the new root and delete it from its original position.
- **Note** we could equally well promote the maximum item in the left subtree to be the new root (and delete it from its original position).



- Here is Python code for the above diagram:

```python
def joinBST(leftT, rightT):
  if isEmptyBT(rightT):
    return leftT
  else:
    (y,t) = splitBST(rightT)
    return mkNodeBT(y, leftT, t)
```

- splitBST will take the right subtree and return a pair of minimum item and the subtree with that item removed.
- This preserves much of the compact nature of the binary search tree.

splitBST Version 1

- We still have choices in defining splitBST
- We could define splitBST by finding the minimum item and then deleting that from the subtree.

```python
def splitBST1(t):
  if isEmptyBT(t):
    raise RuntimeError("splitBST1␣applied␣to␣EmptyBT()")
  elif isEmptyBT(getLeftBT(t)):
    return (getDataBT(t), getRightBT(t))
  else:
    y = minItemBST(getLeftBT(t))
    return (y, mkNodeBT(getDataBT(t),
                        deleteBST(y, getLeftBT(t)),
                        getRightBT(t)))

def minItemBST(t):
  if isEmptyBT(t):
    raise RuntimeError("minItemBST␣applied␣to␣EmptyBT()")
  elif isEmptyBT(getLeftBT(t)):
    return (getDataBT(t))
  else:
    return (minItemBST(getLeftBT(t)))
```

splitBST Final Version

- We can do better than splitBST1
- It is possible to define splitBST using only one traversal of the tree.

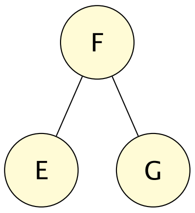
```python
def splitBST(t):
  if isEmptyBT(t):
    raise RuntimeError("splitBST␣applied␣to␣EmptyBT()")
  else:
```

```
731     x = getDataBT(t)
732     t1 = getLeftBT(t)
733     t2 = getRightBT(t)
734     if isEmptyBT(t1):
735       return (x,t2)
736     else:
737       (y,t3) = splitBST(t1)
738       return (y, mkNodeBT(x, t3, t2))
```

Activity 12 Split Trace

- Trace an evaluation of `splitBST(egBSTreeLR)`

splitBST(egBSTreeLR)



```
160 egBSTreeLR = mkNodeBT('F',
161                  mkNodeBT('E',mkEmptyBT(),mkEmptyBT()),
162                  mkNodeBT('G',mkEmptyBT(),mkEmptyBT())
163               )
```

- egBSTreeLR is defined at line 160 on page 53, splitBST is defined at line 727 on page 52

Go to Answer

Answer 12 Split Trace

- Evaluation of `splitBST(egBSTreeLR)`

  Step 1

  `getLeftBT(egBSTreeLR)` is not empty so the `else` clause at line 736 is executed

  Step 2

  A recursive call is made to `splitBST` with argument `getLeftBT(egBSTreeLR)`

  `getLeftBT(getLeftBT(egBSTreeLR))` is empty so the `if` at line 734 returns

  `('E',getRightBT(getLeftBT(egBSTreeLR)))` which is `('E',EmptyBT())`

Answer 12 Split Trace

  Step 3

  The calling function then returns

  `('E', makeBT('F', EmptyBT(), getRightBT(egBSTreeLR)))`

```
  ('E',NodeBT('F',
        EmptyBT(),
        NodeBT('G',EmptyBT(),EmptyBT())))
```

Go to Activity

Activity 13 Join Trace

- Trace an evaluation of

```
joinBST(egBSTreeLL,egBSTreeLR)
```

- egBSTreeLL is defined at line 146 on page 22, joinBST is defined at line 696 on page 52

Go to Answer

Answer 13 Join Trace

- Evaluation of `joinBST(egBSTreeLL,egBSTreeLR)`

Step 1

The second argument to `joinBST` is not the empty tree so the `else` clause at line 699 — this invokes `splitBST(egBSTreeLR)`

Step 2

From the previous activity, `splitBST(egBSTreeLR)` returns

```
('E', makeBT('F', EmptyBT(), getRightBT(egBSTreeLR)))
```

Answer 13 Join Trace

Step 3

Finally, the `return` statement returns

```
NodeBT('E',
  NodeBT('B',
    NodeBT('A', EmptyBT(), EmptyBT()),
    NodeBT('C', EmptyBT(), EmptyBT())),
  NodeBT('F',
    EmptyBT(),
    NodeBT('G', EmptyBT(), EmptyBT())))
```

Go to Activity

Activity 14 Delete Trace

- Trace an evaluation of

```
deleteBST('D',egBSTreeL)
```

- egBSTreeL is defined at line 122 on page 21, deleteBST is defined at line 627 on page 48

Go to Answer

Answer 14 Delete Trace

- Evaluation of `deleteBST('D',egBSTreeL)`

Step 1

The second argument of `deleteBST` is not the empty tree so the `else` clause at line 630 is executed.

Step 2

The first argument of `deleteBST` is `'D'` which is equal to the item at the root of the tree which is the second argument, so the `else` clause at line 638 is executed

Step 3

This evaluates to `joinBST(egBSTreeLL,egBSTreeLR)` — see previous activity

Go to Activity

- As we noted earlier, on average the height of a binary search tree is $O(\log n)$ where n is the number of nodes in the tree.
- However in the worst case the height is $O(n)$ and this will affect the performance of searches.
- However it is possible to construct variants of binary search trees which have $O(\log n)$ performance in both average and worst cases
- In the next section we will consider one approach.

ToC

Commentary 4

4 Height Balanced Trees

- Binary search trees with the *height balanced* property
- Also called AVL trees
- Inserting a node
- AVL transformations
- Local changes preserve global AVL property
- Deletion
- AVL trees application: representing sets (advanced topic)
- Note: Haskell uses the same ideas but with *size* balanced trees
- Python uses something like dictionaries to implement sets using hashtables

ToC

6 Height Balanced Trees

- Binary search trees have the problem that in the worst case the complexity of a search could be $O(n)$ and maintaining a complete tree during insertions and deletions involves too much restructuring.
- A solution is to keep the tree *balanced* so that access time is still $O(\log n)$ in both the average and worst cases.
- The essential approach is to have some *local* transformations involving only a few nodes to keep the tree *height balanced*.
- We shall consider one approach called *AVL Trees*, named after the Russian inventors G M Adelson-Velskii and E M Landis (Adelson-Velskii and Landis, 1962).

- *AVL Trees* are Binary Search Trees with the property that for every subtree the heights of the trees differs by at most 1 (the *balance factor*).
- AVL trees require an extra couple of functions to maintain the AVL property on each insertion or deletion.

6.1 AVL Trees and Functions

AVL Tree Data Type

- As with the Binary Search Tree, we shall use a union of named tuples to represent the data type for an AVL Tree.
- Note that we store the height of a tree in the node
- This is essential to avoid lots of tree traversals to re-calculate balance factors

```python
# AVL Tree Data Type

# from collections import namedtuple

EmptyABT = namedtuple('EmptyABT',[])

NodeABT = (namedtuple('NodeABT',
            ['heightABT','dataABT','leftABT','rightABT']))

# Tree type --- Augmented Binary Tree

# from typing import TypeVar,Union,NewType

# T = TypeVar('T')
ABTree = NewType('ABTree',Union[EmptyABT,NodeABT]).
```

AVL Tree Operations

```python
# AVL Tree Operations

def mkEmptyABT() -> ABTree :
  return EmptyABT()

def mkNodeABT(x: T,t1: ABTree,t2: ABTree) -> ABTree :
  h = 1 + max(getHeightABT(t1),getHeightABT(t2))
  return NodeABT(h,x,t1,t2)

def isEmptyABT(t: ABTree) -> bool :
  return t == EmptyABT()
```

```python
def getHeightABT(t: ABTree) -> int :
  if isEmptyABT(t) :
    return 0
  else:
    return t.heightABT

def getDataABT(t: ABTree) -> T :
  if isEmptyABT(t) :
    raise RuntimeError("getDataABT␣applied␣to␣EmptyABT()")
  else:
    return t.dataABT

def getLeftABT(t: ABTree) -> ABTree :
  if isEmptyABT(t) :
    raise RuntimeError("getLeftABT␣applied␣to␣EmptyABT()")
  else:
    return t.leftABT

def getRightABT(t: ABTree) -> ABTree :
  if isEmptyABT(t) :
    raise RuntimeError("getRightABT␣applied␣to␣EmptyABT()")
  else:
    return t.rightABT
```

AVL Trees Property Functions

```
def isBSABTree(t):
  return orderedList(inOrderABT(t))

def inOrderABT(t):
  if isEmptyABT(t):
    return []
  else:
    return (inOrderABT(getLeftABT(t)) + [getDataABT(t)]
            + inOrderABT(getRightABT(t)))
```

```
def convertBTtoABT(t):
  if isEmptyBT(t):
    return mkEmptyABT()
  else:
    leftABT = convertBTtoABT(getLeftBT(t))
    rightABT = convertBTtoABT(getRightBT(t))
    return mkNodeABT(getDataBT(t), leftABT, rightABT)
```

```
def balFactorABT(t):
  if isEmptyABT(t):
    return 0
  else:
    return (getHeightABT(getLeftABT(t))
            - getHeightABT(getRightABT(t)))
```

```
def hasAVLpropABT(t):
  if isEmptyABT(t):
    return True
  else:
    return (abs(balFactorABT(t)) <= 1
            and hasAVLpropABT(getLeftABT(t))
            and hasAVLpropABT(getRightABT(t)))
```

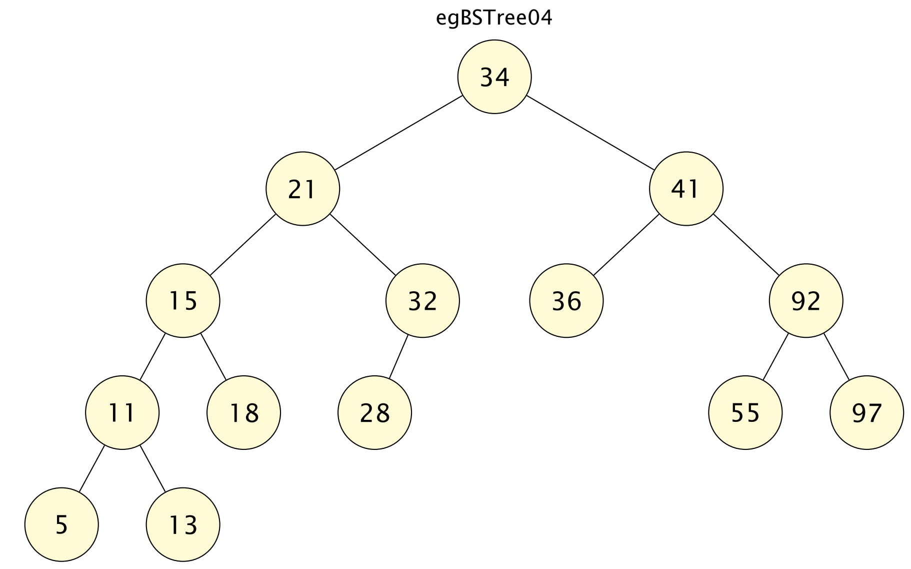
```
def isAVLABTree(t):
  return (isBSABTree(t) and hasAVLpropABT(t))
```

- Some texts define the height of a singleton node to be zero — just subtract one from the height as defined here.
- Some texts do not use empty trees — so where these notes might say a singleton nodes has an element and two empty subtrees, some texts might say a singleton node has no subtrees
- Some texts define the height of a subtree differently to the height of a tree or define a subtree differently to here.
- Some texts define the balance factor as the absolute value or the height of the right sub tree minus the height of the left sub tree
- In all cases be aware that you have choices in the exact definition of some terms but the ideas will be the same.

6.2 Example AVL Trees

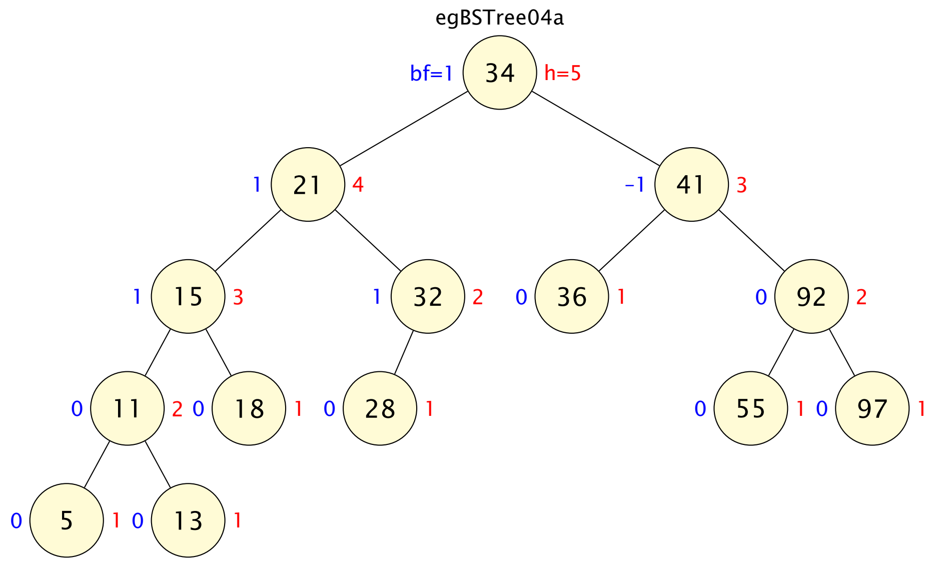
Activity 15 Height and Balance Factor

- For the following diagram of a binary search tree, egBSTree04, add the height and balance factor for each node.



Go to Answer

Answer 15 Height and Balance Factor



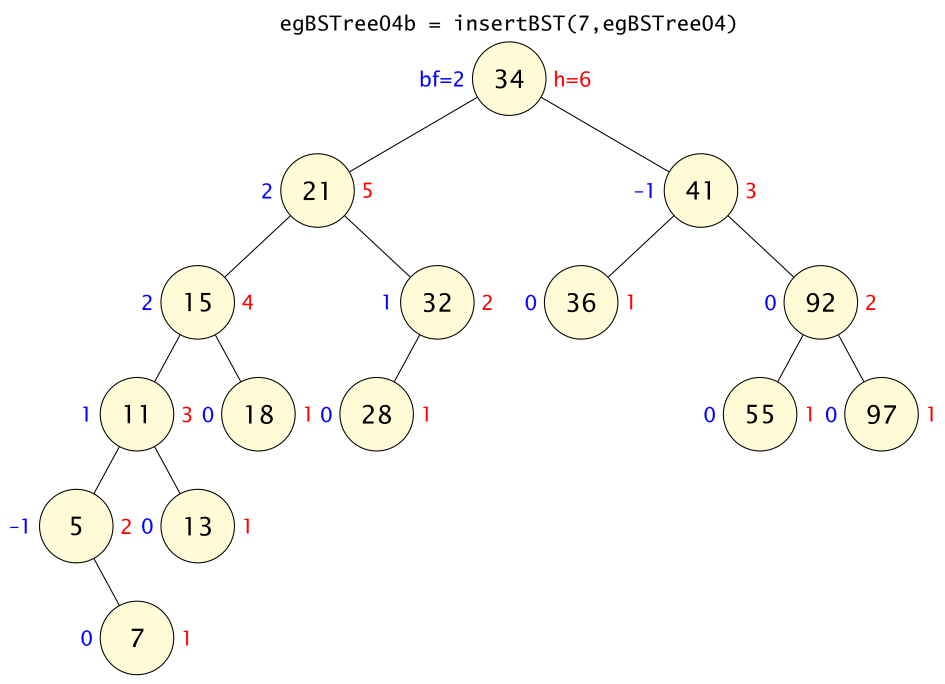
Go to Activity

Activity 16 Add Item LL

- Add the item with key 7 to the tree, egNSTree04, and recalculate the heights and balance factors
- Identify the lowest node in the tree which is out of balance.

Go to Answer

Answer 16 Add Item LL



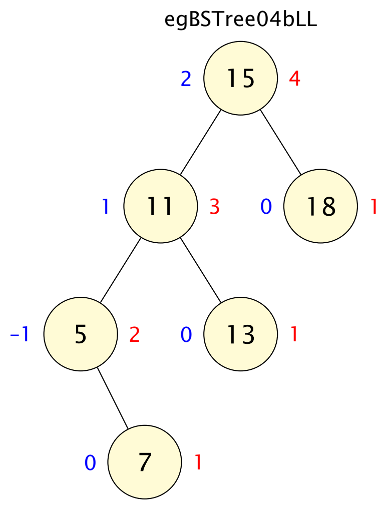
- Lowest node which is out of balance is node with key 15

Go to Activity

6.3 AVL Tree Transformations

- Since the subtree of egBSTree04b = insertBST(7,egBSTree04) at node with key 15 is the part of the tree out of balance we shall focus on that

```
NodeABT(4, 15,
  NodeABT(3, 11,
    NodeABT(2, 5,
      EmptyABT(),
      NodeABT(1, 7, EmptyABT(), EmptyABT())),
    NodeABT(1, 13, EmptyABT(), EmptyABT())),
  NodeABT(1, 18, EmptyABT(), EmptyABT()))
```

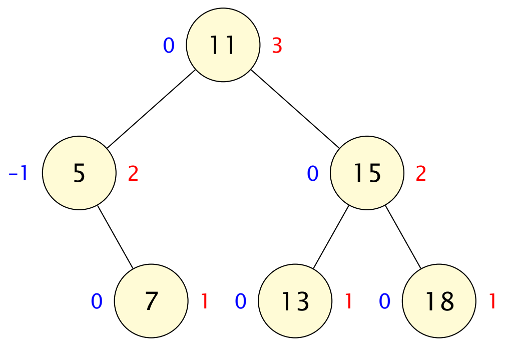


- We can make this tree balanced by
- Make the subtree with root 15 the right child of 11
- Make the subtree with root 13 the left child of 15
- Leave the subtree with root 5 as the left child of 11

- Make the new subtree with root 11 the child of wherever the original subtree with root 15 was (the left child of 21)
- This results in the following tree.

```
NodeABT(3, 11,
  NodeABT(2, 5,
    EmptyABT(),
    NodeABT(1, 7, EmptyABT(), EmptyABT())),
  NodeABT(2, 15,
    NodeABT(1, 13, EmptyABT(), EmptyABT()),
    NodeABT(1, 18, EmptyABT(), EmptyABT())))
```

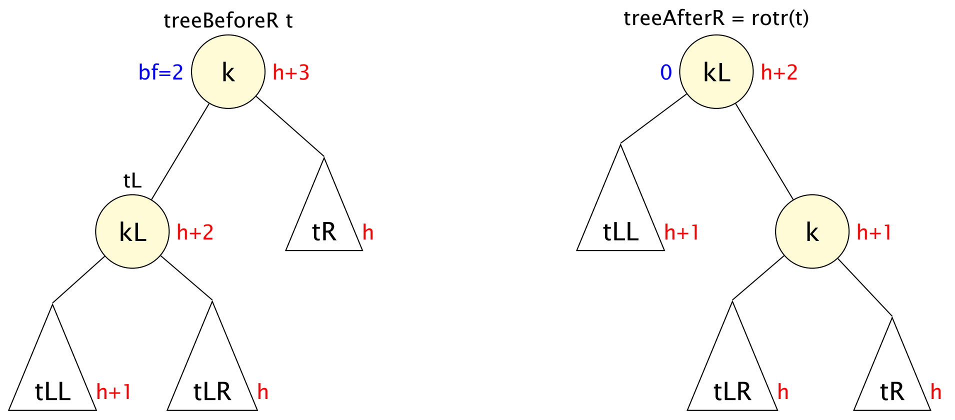
egBSTree04bLLb



- This transformation is an instance of what is called a *right rotation*
- Here is Python code that implements it.

```
def rotr(t):
  k = getDataABT(t)
  kL = getDataABT(getLeftABT(t))
  leftLeftT = getLeftABT(getLeftABT(t))
  leftRightT = getRightABT(getLeftABT(t))
  rightT = getRightABT(t)
  return (mkNodeABT(kL,
                    leftLeftT,
                    mkNodeABT(k, leftRightT, rightT)))
```

Case LL Heavy — Right Rotation

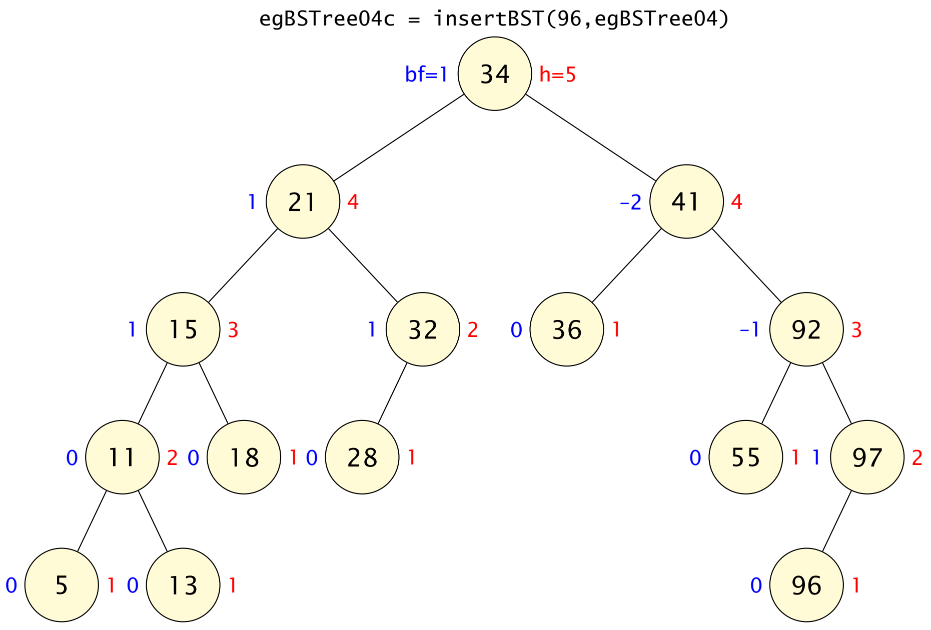


Activity 17 Add Item RR

- Consider egBSTree04 again (defined in Activity 15 on page 57) — now add node with key 96 and recalculate the heights and balance factors

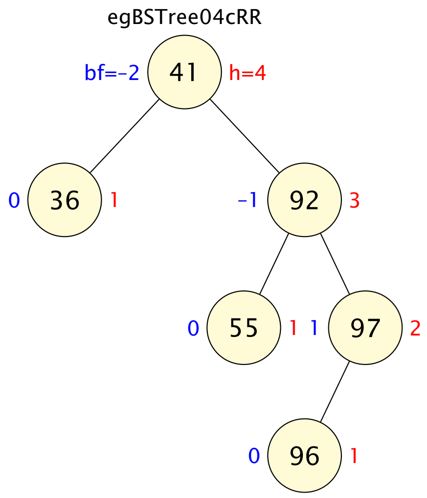
Go to Answer

Answer 17 Add Item RR



Go to Activity

- The subtree at node 41 is now unbalanced with the addition of the node with key 96 to the right subtree of the right subtree.



Activity 18 Rebalance RR

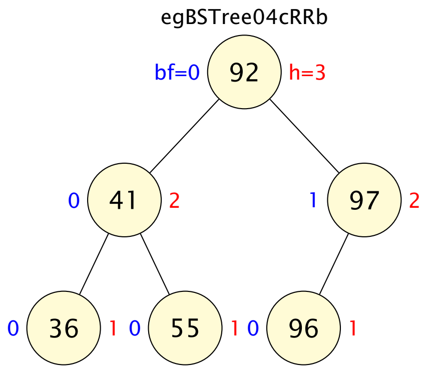
- This is similar to the previous example but on the right side
- Describe how this can be rebalanced using a mirror image local transformation.

Go to Answer

Answer 18 Rebalance RR

- We can make this tree balanced by:
- Make the subtree with root 41 the left child of 92
- Make the subtree with root 55 the right child of 41
- Leave the subtree with root 97 as the right child of 92
- Make the new subtree with root 92 the child of wherever the original subtree with root 41 was (the right child of 34)

```
  NodeABT(3, 92,
    NodeABT(2, 41,
      NodeABT(1, 36, EmptyABT(), EmptyABT()),
      NodeABT(1, 55, EmptyABT(), EmptyABT())),
    NodeABT(2, 97,
      NodeABT(1, 96, EmptyABT(), EmptyABT()),
      EmptyABT()))
```



- The transformation is called a *left rotation*

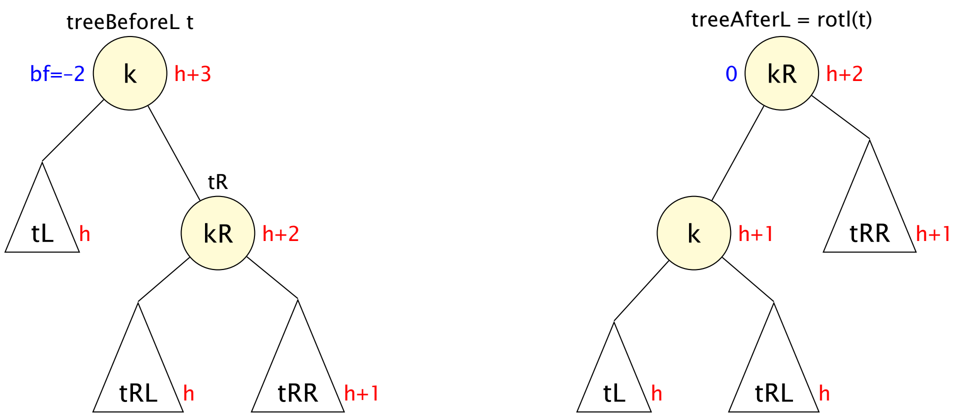
Go to Activity

- The transformation (given in the answer) is an instance of what is called a *left rotation*
- Here is the Python code that implements it.

```
def rotl(t):
  k = getDataABT(t)
  kR = getDataABT(getRightABT(t))
  rightLeftT = getLeftABT(getRightABT(t))
  rightRightT = getRightABT(getRightABT(t))
  leftT = t.leftABT
  return (mkNodeABT(kR,
                    mkNodeABT(k, leftT, rightLeftT),
                    rightRightT))
```

- This is a mirror image of the *right rotation* as you can see from the two diagrams describing it below.

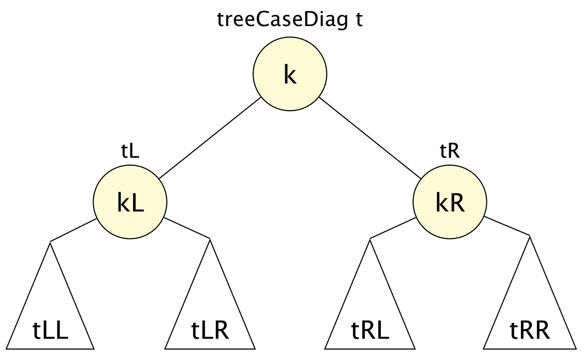
Case RR Heavy — Left Rotation



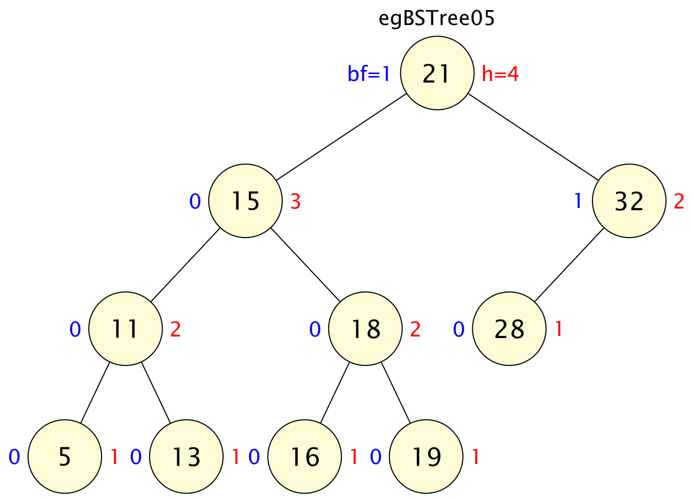
AVL Tree Transformations — Insight

- The functions for insertion and deletion of an item in an AVL tree will be the same as a Binary Search tree except

- When we construct a new tree we must maintain the AVL property via a function `makeAVLTree` (line 865 on page 69) not just `makeABTree` (line 764 on page 56). (some texts call this rebalancing or something similar)
- What we know is that the original tree must be a properly formed AVL tree and that the insertion or deletion of one item can alter the height of any subtree by at most 1.
- Hence we can implement `makeAVLTree(x, leftT, rightT)` assuming that `leftT` and `rightT` are both AVL trees whose heights differ by at most 2.
- We proceed by analysing each possible case and provide a manipulation of the tree for each case. Consider the diagram below:



- Our right and left rotation functions, `rotr` and `rotl` have dealt with the cases where the subsubtrees LL and RR had increased by one caused the balance to go outside the permitted range.
- We now have to investigate cases where the subsubtrees LR or RL become heavy.
- Below is an example, `egBSTree05`

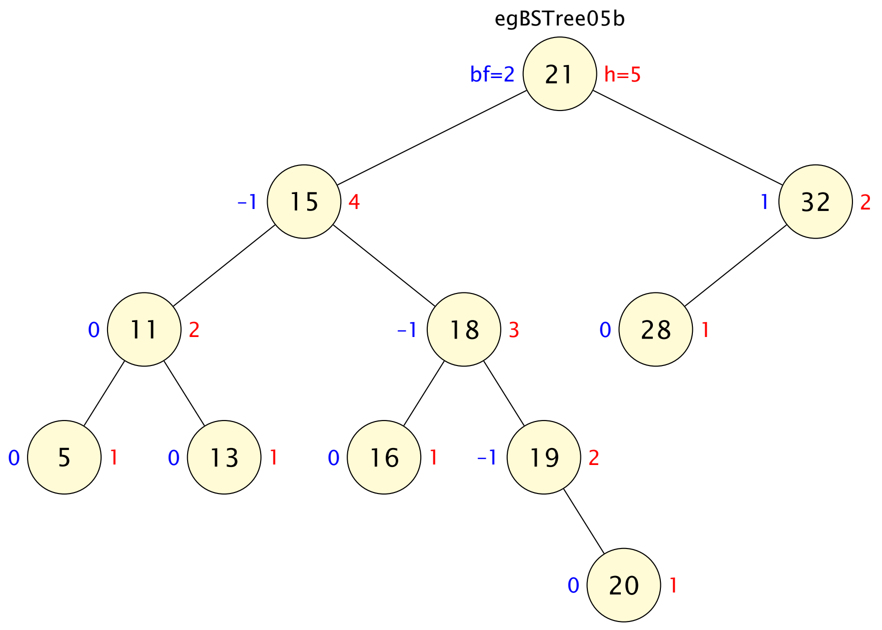


Activity 19 egBSTree05 Add Item LR 1

- Add the item with key 20 to the tree and recalculate the heights and balance factors
- Identify the lowest node in the tree which is out of balance.

Go to Answer

Answer 19 egBSTree05 Add Item LR 1



- The node with key 21 has balance factor 2 and is the lowest node out of balance.

Go to Activity

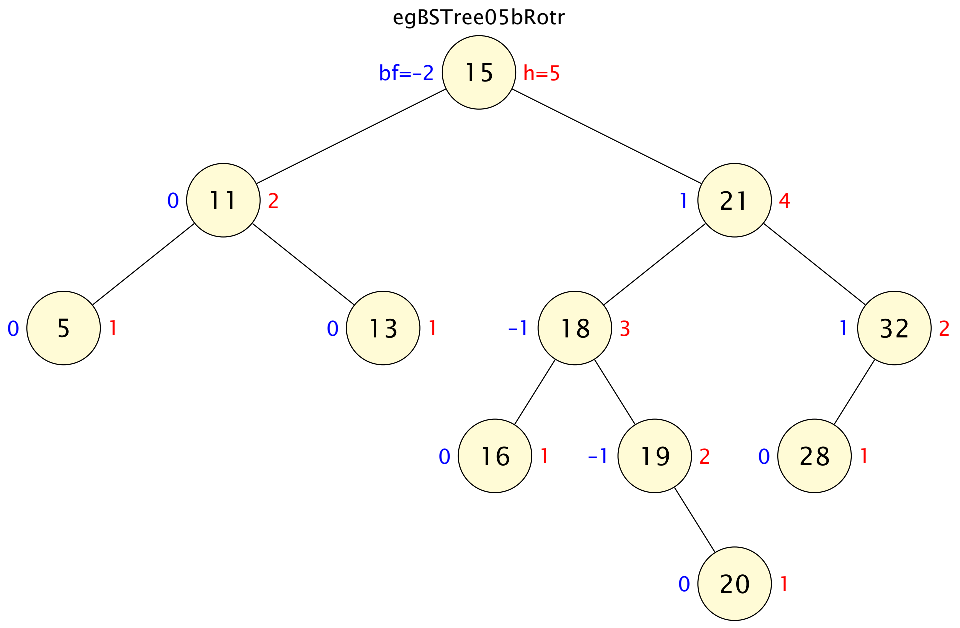
Activity 20 Add Item LR 2

- Given the resulting tree from Self-assessment activity 19, does a right rotation around the lowest node which is out of balance bring it back to balance ?

Go to Answer

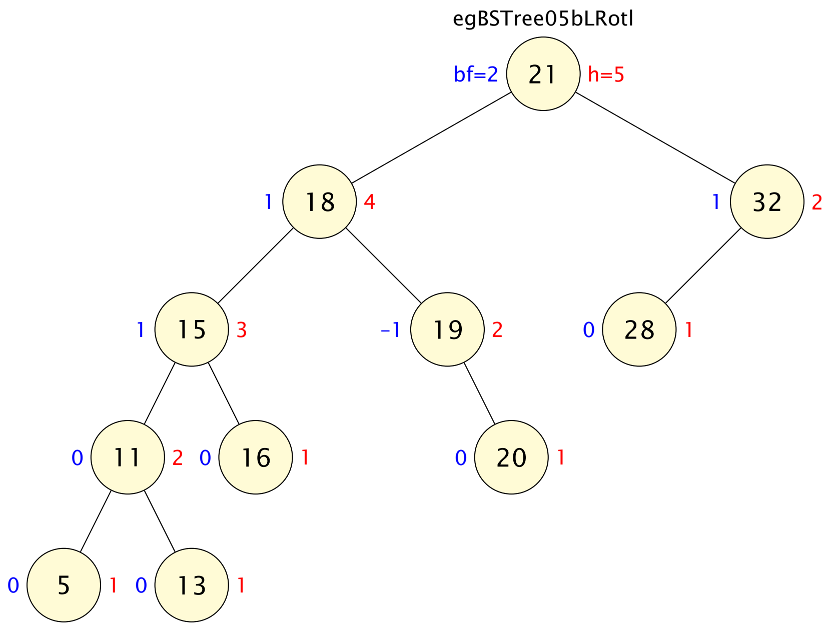
Answer 20 Add Item LR 2

- Here is the result of a right rotation around node with key 21

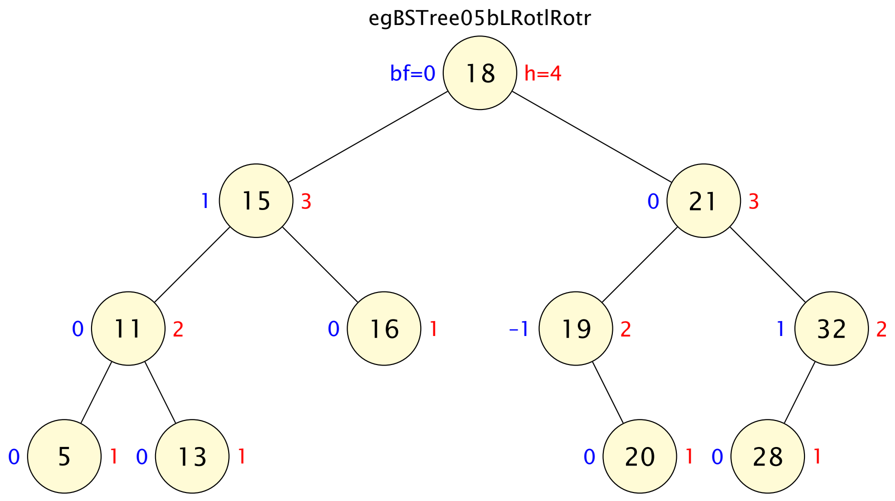


Go to Activity

- This has just switched the balance factor of the root of the tree from 2 to -2 so we have to do something else.
- The *Eureka* step is realising that we can break up the problematic subtree under node 18 by doing a left rotation around node 15 — this produces the following tree.

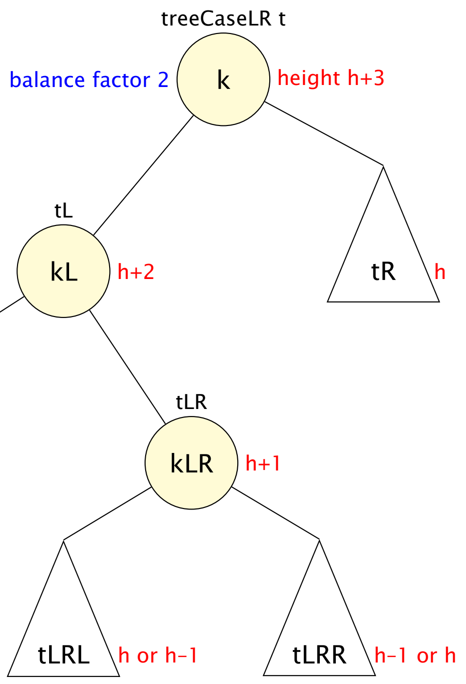


- Notice that this has converted a tree which was LR heavy to one where it is LL heavy — so we can now use a right rotation on the tree rooted at 21 to get the following:

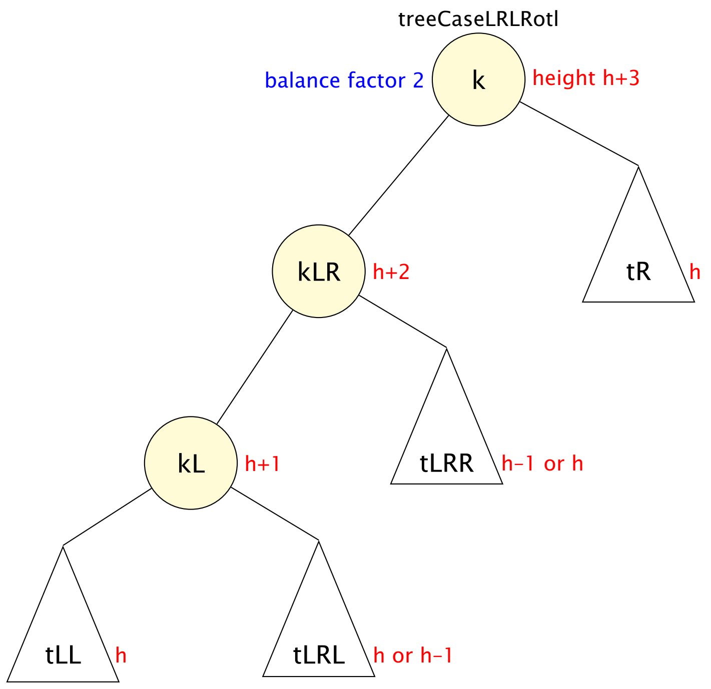


- We now have a balanced tree — but were we just lucky or have we found a general rule ? Here are diagrams of the double rotation to show it works in general:

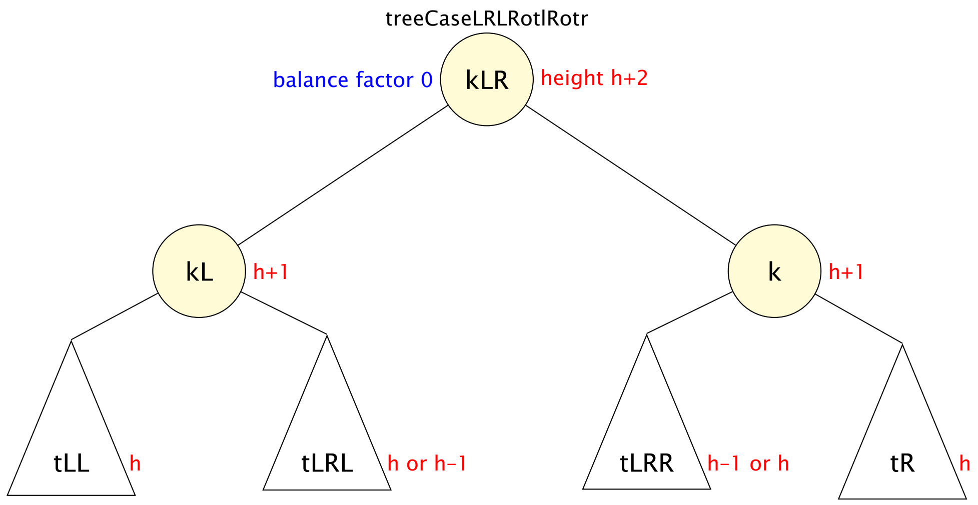
Case LR subsubtree heavy



Case LR — Step (A) Rotate Left about kL



Case LR — Step (B) Rotate Right about k

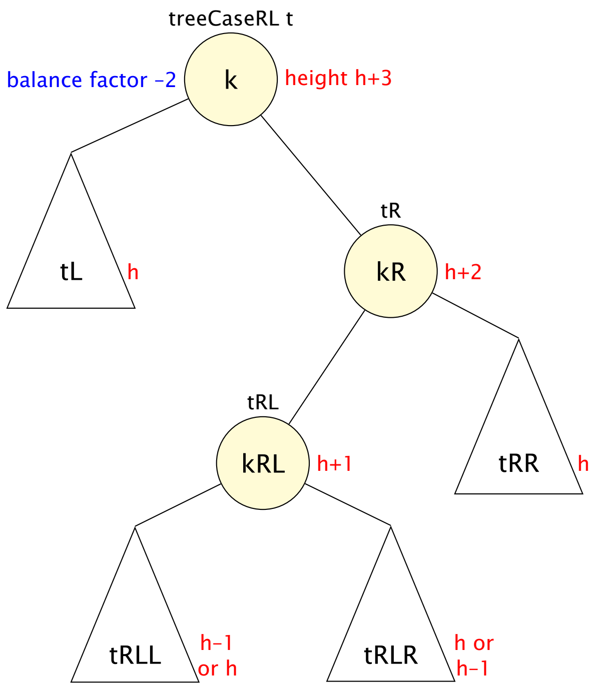


Activity 21 Case RL Heavy

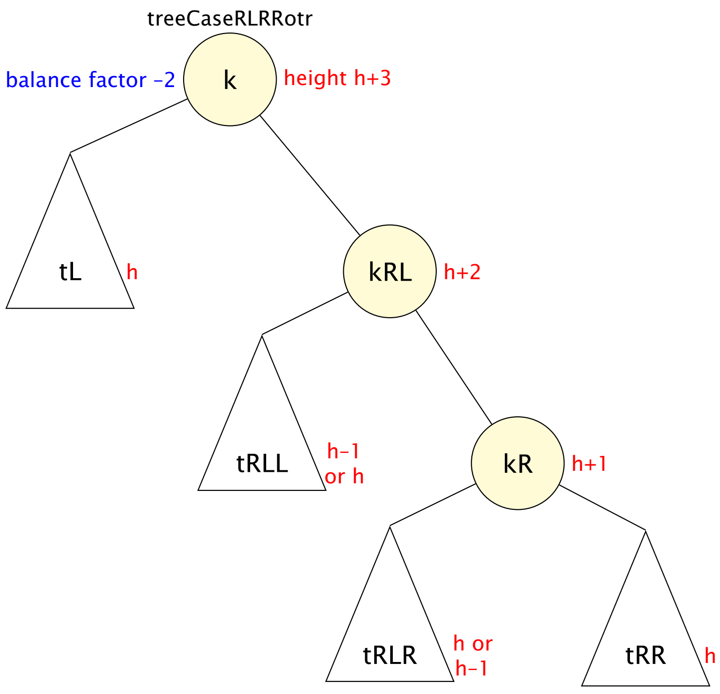
- Draw the equivalent diagram for the final case where subsubtree tRL is heavy
- Note that this must be the mirror image of the tLR case

Go to Answer

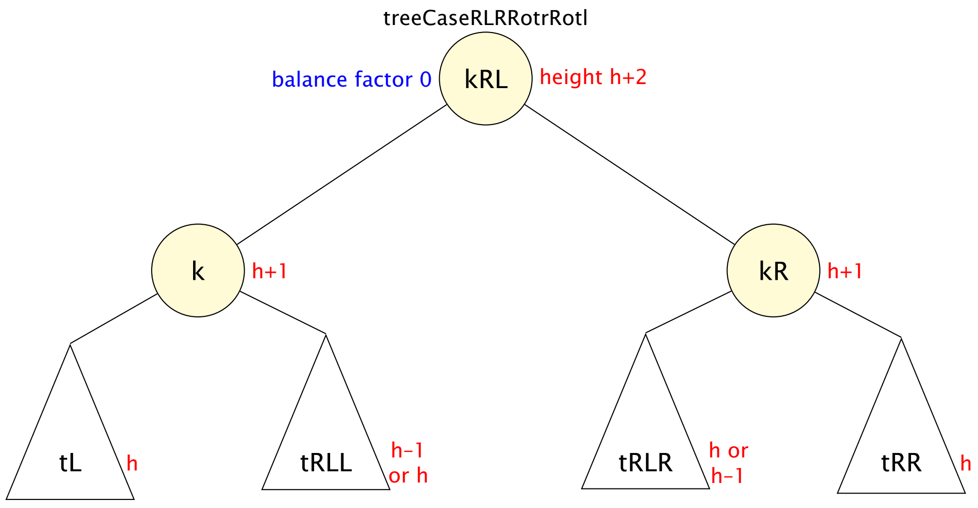
Answer 21 Case RL Heavy



Answer 21 Case RL Heavy — Step (A) Rotate Right about kR



Answer 21 Case RL Heavy — Step (B) Rotate Left about k



Go to Activity

6.4 AVL Trees — The makeAVLTree Function

- The makeAVLTree takes an item, x, two subtrees, leftT, rightT and returns a new augmented binary tree
- It would be the same as makeABTree except it has to do the appropriate transformation if the new tree would be out of balance.
- We will only ever use makeAVLTree when inserting or deleting an item in a valid AVL tree
- So we know from our insight above that the heights of leftT and rightT can differ by at most 2 after insertion/deletion
- Hence we consider each case in turn using the transformations we have developed above.

Case 1 LL Heavy

```
(getHeightABT(leftT) - getHeightABT(rightT) = 2
 and balFactorABT(leftT) >= 0)
```

- Do a right rotation of the tree formed from `makeABTree(x, leftT, rightT)`

Case 2 LR Heavy

```
(getHeightABT(leftL) - getHeightABT(rightT) = 2
 and balFactorABT(leftT) == -1)
```

- Do a left rotation of `leftT`
- Do a right rotation of the tree formed from `makeABTree(x, rotl(leftT), rightT)`

Case 3 RL Heavy

```
(getHeightABT(leftL) - getHeightABT(rightT) = -2
 and balFactorABT(rightT) == 1)
```

- Do a right rotation of `rightT`
- Do a left rotation of the tree formed from `makeABTree(x, leftT, rotr(rightT))`

Case 4 RR Heavy

```
(getHeightABT(leftL) - getHeightABT(rightT) = -2
 and balFactorABT(rightT) <= 0)
```

- Do a left rotation of the tree formed from `makeABTree(x, leftT, rightT)`

Case 5 Otherwise

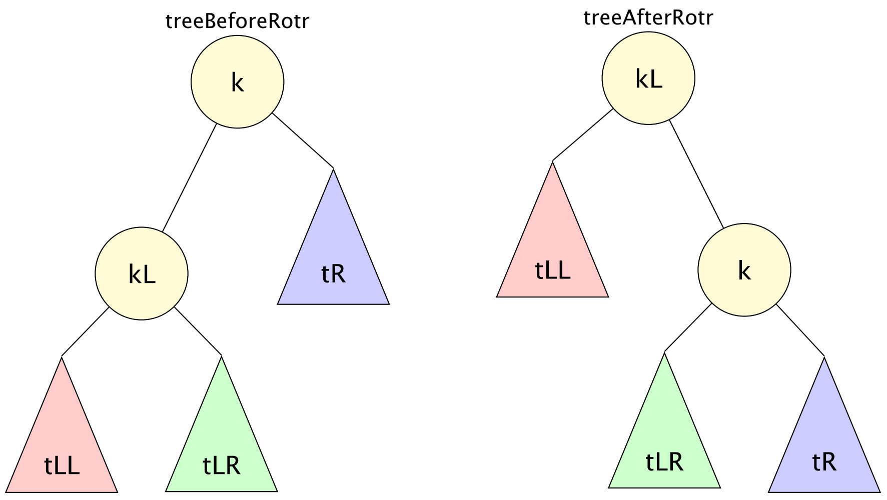
- Just use `makeABTree(x, leftT, rightT)`
- No transformations required

`makeAVLTree` Function — Python

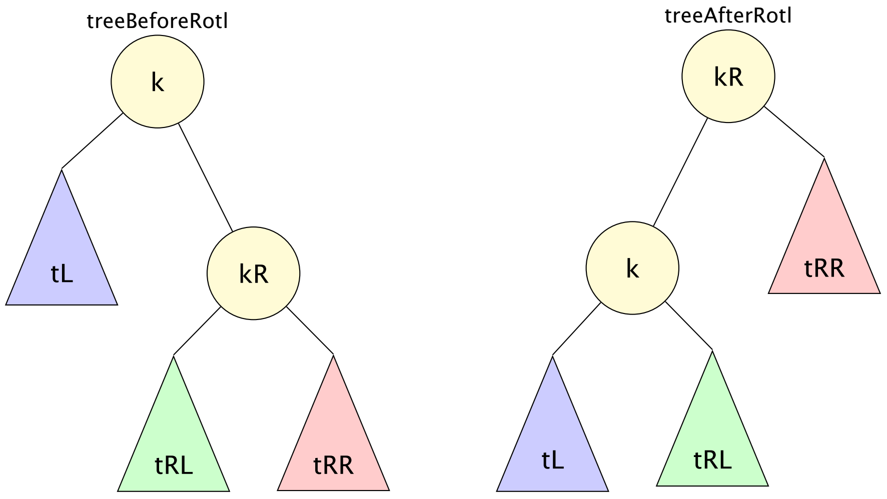
```
def makeAVLTree(x, leftT, rightT):
  hL = getHeightABT(leftT)
  hR = getHeightABT(rightT)
  if (hR + 1 < hL) and (balFactorABT(leftT) >= 0):
    return rotr(mkNodeABT(x, leftT, rightT))
  elif (hR + 1 < hL):
    return rotr(mkNodeABT(x, (rotl(leftT)),rightT))
  elif (hL + 1 < hR) and (balFactorABT(rightT) > 0):
    return rotl(mkNodeABT(x, leftT, rotr(rightT)))
  elif (hL + 1 < hR):
    return rotl(mkNodeABT(x, leftT, rightT))
  else:
    return mkNodeABT(x, leftT, rightT)
```

- This section has been quite long but most of the space has been occupied with diagrams
- Some *implementations* can look quite tricky since they may be trying to avoid recursion or manipulate the data structures
- We will discuss efficiency and recursion removal in a later section.
- Here are diagrams of the two rotate functions to emphasise that they are really quite simple.

Rotate Right



Rotate Left



6.4.1 Comparison with Storing Balance Factors

- Some texts implement AVL Trees by storing balance factors at the nodes rather than the heights (Miller and Ranum, 2011, Section 6.8.2, page 290)
- The Miller and Ranum explanation of updating the balance factors after a right or left rotation refer to diagrams similar to rotate right and rotate left (page 70)
- This note translates the Miller & Ranum notation to the notation used in these diagrams
- Both approaches have performance $O(\log n)$ but have differences in detail

Right rotation

- New and old balance factors of node k
- Using pseudo-code:

```
newBal(k) = height(tLR) - height(tR)
oldBal(k) = oldHeight(kL) - height(tR)
          = (1 + max(height(tLL), height(tLR)))
            - height(tR)

newBal(k) - oldBal(k)
```

```
  = (height(tLR) - height(tR))
    - ((1 + max(height(tLL), height(tLR)))
       - height(tR))
  = height(tLR)
    - 1 - max(height(tLL), height(tLR))
  = height(tLR)
    - 1 + min(-height(tLL), -height(tLR))
  = min(height(tLR) - height(tLL)
       , height(tLR) - height(tLR)) - 1
  = min(-oldBal(kL), 0) - 1
    since -max(a, b)     = min(-a,-b)
           min(a,b) + c = min(a+c, b+c)
```

- New and old balance factors of node kL

```
newBal(kL) = height(tLL) - newHeight(k)
           = height(tLL)
             - (1 + max(height(tLR), height(tR)))
oldBal(kL) = height(tLL) - height(tLR)

newBal(kL) - oldBal(kL)
  = (height(tLL)
    - (1 + max(height(tLR), height(tR))))
       - (height(tLL) - height(tLR))
  = height(tLR)
    - 1 - max(height(tLR), height(tR))
  = height(tLR)
    - 1 + min(-height(tLR), -height(tR))
  = min(height(tLR) - height(tLR)
       , height(tLR) - height(tR)) - 1
  = min(0, newBal(k)) - 1
```

- Right rotation: New and old balance factors of nodes k and kR

```
newBal(k)
  = oldBal(k) + min(-oldBal(kL), 0) - 1

newBal(kL)
  = oldBal(kL) + min(0, newBal(k)) - 1
```

- This fits with the right rotation diagrams annotated with heights and balance factors on page 60,

```
oldBal(k)  = +2
oldBal(kL) = +1
newBal(k)  =  0 = +2 + min(-1, 0) - 1
newBal(kL) =  0 = +1 + min(0, 0) - 1
```

Left rotation

- New and old balance factors of node k
- Using pseudo-code:

```
newBal(k) = height(tL) - height(tRL)
oldBal(k) = height(tL) - oldHeight(kR)
          = height(tL)
            - (1 + max(height(tRL), height(tRR)))

newBal(k) - oldBal(k)
  = (height(tL) - height(tRL))
    - (height(tL)
       - (1 + max(height(tRL), height(tRR))))
  = 1 + max(height(tRL), height(tRR)) - height(tRL)
  = 1 + max(height(tRL) - height(tRL)
           , height(tRR) - height(tRL))
  = 1 + max(0, -oldBal(kR))
  = 1 - min(0, oldBal(kR))
    since max(-a, -b)  = -min(a,b)
          max(a,b) - c = max(a-c, b-c)
```

- New and old balance factors of node kR

```
newBal(kR) = newHeight(k) - height(tRR)
           = (1 + max(height(tL), height(tRL)))
             - height(tRR)
oldBal(kR) = height(tRL) - height(tRR)

newBal(kR) - oldBal(kR)
  = ((1 + max(height(tL), height(tRL)))
    - height(tRR))
       - (height(tRL) - height(tRR))
  = 1 + max(height(tL), height(tRL)) - height(tRL)
  = 1 + max(height(tL) - height(tRL)
           , height(tRL) - height(tRL))
  = 1 + max(newBal(k), 0)
```

- Left rotation: New and old balance factors of nodes k and kR

```
newBal(k)
  = oldBal(k) + 1 - min(0, oldBal(kR))

newBal(kR)
  = oldBal(kR) + 1 + max(newBal(k), 0)
```

- This fits with the left rotation diagrams annotated with heights and balance factors on page 62,

```
oldBal(k)  = -2
oldBal(kR) = -1
newBal(k)  =  0 = -2 + 1 - min(0,-1)
newBal(kR) =  0 = -1 + 1 + max(0, 0)
```

ToC

6.5 AVL Tree Insertion and Deletion

- The insertion and deletion functions are the same as for Binary Search Trees except we have to use makeAVLTree to make a tree unless we really know that the AVL property will be preserved.

```python
def insertAVLT(x,t):
  if isEmptyABT(t):
    return mkNodeABT(x, mkEmptyABT(), mkEmptyABT())
  else:
    y = getDataABT(t)
    leftT = getLeftABT(t)
    rightT = getRightABT(t)
    if x < y:
      return makeAVLTree(y, insertAVLT(x, leftT), rightT)
    elif x > y:
      return makeAVLTree(y, leftT, insertAVLT(x, rightT))
    else:
      return t
```

```python
def insertListAVLT(xs,t):
  if xs == []:
    return t
  else:
    return insertListAVLT(xs[1:], (insertAVLT(xs[0],t)))
```

```python
def deleteAVLT(x,t):
  if isEmptyABT(t):
    return mkEmptyABT()
  else:
    y = getDataABT(t)
    leftT = getLeftABT(t)
    rightT = getRightABT(t)
    if x < y:
      return makeAVLTree(y, deleteAVLT(x, leftT), rightT)
    elif x > y:
      return makeAVLTree(y, leftT, deleteAVLT(x, rightT))
    else:
      return joinAVLT(leftT, rightT)
```

```
def joinAVLT(leftT, rightT):
  if isEmptyABT(rightT):
    return leftT
  else:
    (y,t) = splitAVLT(rightT)
    return makeAVLTree(y, leftT, t)
```

```
def splitAVLT(t):
  if isEmptyABT(t):
    raise RuntimeError("splitAVLT␣applied␣to␣EmptyABT()")
  else:
    x = getDataABT(t)
    t1 = getLeftABT(t)
    t2 = getRightABT(t)
    if isEmptyABT(t1):
      return (x,t2)
    else:
      (y,t3) = splitAVLT(t1)
      return (y, makeAVLTree(x, t3, t2))
```

Activity 22 Insert Lists and Delete Items

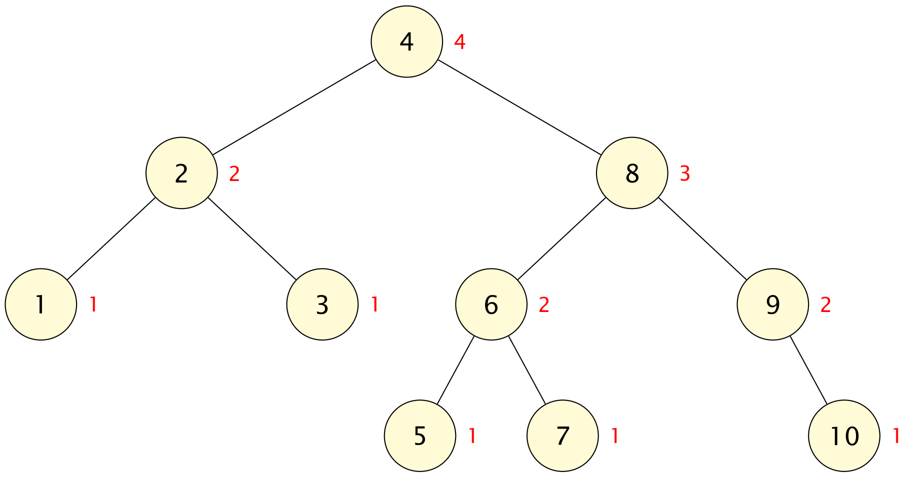
- Draw the AVL Trees resulting from inserting the following lists of items into an empty tree one by one in order given — do the insertions by hand following the AVL insertion algorithm — you can use the Python code to check your answers
  1. [1,2,3,4,5,6,7,8,9,10]
  2. [10,9,8,7,6,5,4,3,2,1]
  3. [68,88,61,89,94,50,4,76,66,82,99]
- For each of the previous trees, show the result when the fourth item inserted is deleted
- The insertAVLT function is defined at line 881, page 72 (Python), the deleteAVLT function is defined at line 901, page 72 (Python),
- insertListAVLT is defined at line 895, page 72 (Python),

Go to Answer

Answer 22 Insert Lists and Delete Items

```
  listQ1a = [1,2,3,4,5,6,7,8,9,10]
  exsAVLInsDelQ1a = insertListAVLT(listQ1a, EmptyABT())
```

exsAVLInsDelQ1a



Answer 22 Insert Lists and Delete Items

- Here is the Python representation of the resulting AVL tree

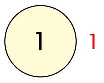
```
exsAVLInsDelQ1aAns \
 = (NodeABT(4, 4,
      NodeABT(2, 2,
        NodeABT(1, 1, EmptyABT(), EmptyABT()),
        NodeABT(1, 3, EmptyABT(), EmptyABT())),
      NodeABT(3, 8,
        NodeABT(2, 6,
          NodeABT(1, 5, EmptyABT(), EmptyABT()),
          NodeABT(1, 7, EmptyABT(), EmptyABT())),
        NodeABT(2, 9,
          EmptyABT(),
          NodeABT(1, 10, EmptyABT(), EmptyABT())))))
```

Answer 22 Insert Lists and Delete Items

- Here are the insertions done one by one with separate diagrams

```
exsAVLInsDelQ1a01 \
 = insertListAVLT(listQ1a[:1], EmptyABT())
```

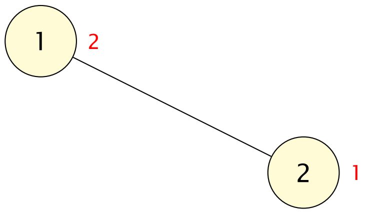
exsAVLInsDelQ1a01



Answer 22 Insert Lists and Delete Items

```
exsAVLInsDelQ1a02 \
 = insertListAVLT(listQ1a[:2], EmptyABT())
```

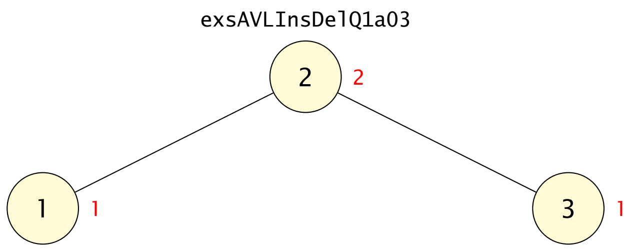
exsAVLInsDelQ1a02



Answer 22 Insert Lists and Delete Items

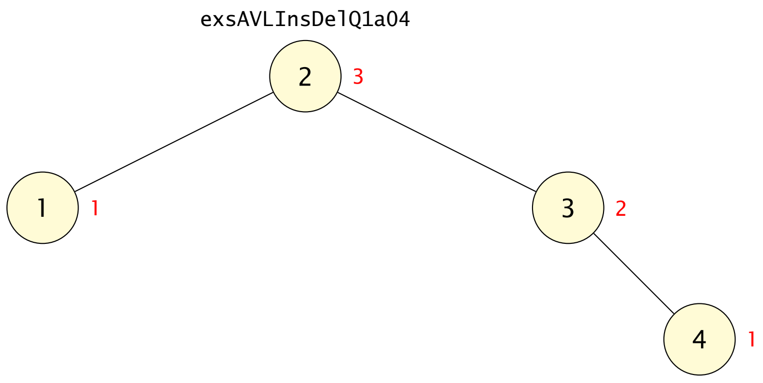
```
exsAVLInsDelQ1a03 \
 = insertListAVLT(listQ1a[:3], EmptyABT())
```

Left rotation about 1



Answer 22 Insert Lists and Delete Items

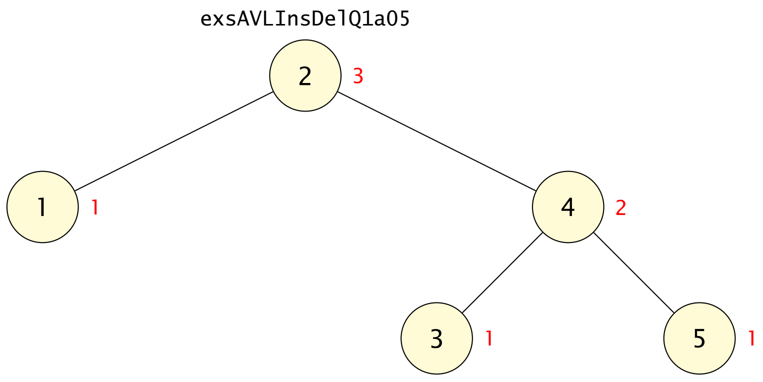
```
exsAVLInsDelQ1a04 \
 = insertListAVLT(listQ1a[:4], EmptyABT())
```



Answer 22 Insert Lists and Delete Items

```
exsAVLInsDelQ1a05 \
 = insertListAVLT(listQ1a[:5], EmptyABT())
```

Left rotation about 3

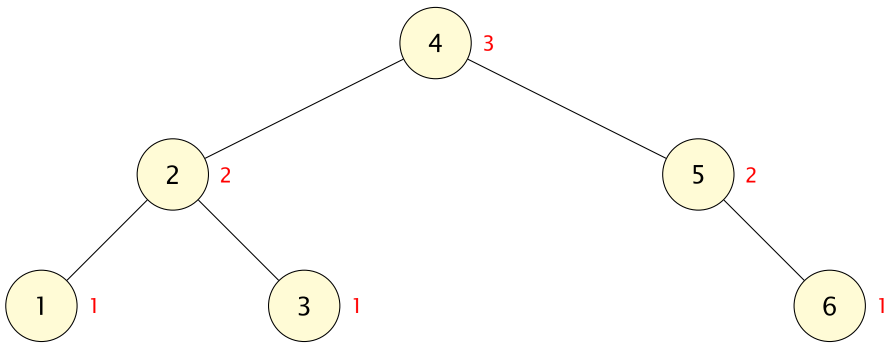


Answer 22 Insert Lists and Delete Items

```
exsAVLInsDelQ1a06 \
 = insertListAVLT(listQ1a[:6], EmptyABT())
```

Left rotation about 2

exsAVLInsDelQ1a06

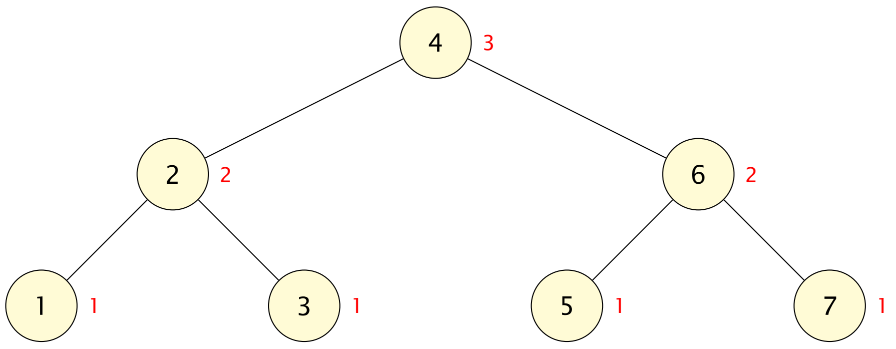


Answer 22 Insert Lists and Delete Items

```
exsAVLInsDelQ1a07 \
 = insertListAVLT(listQ1a[:7], EmptyABT())
```

Left rotation about 5

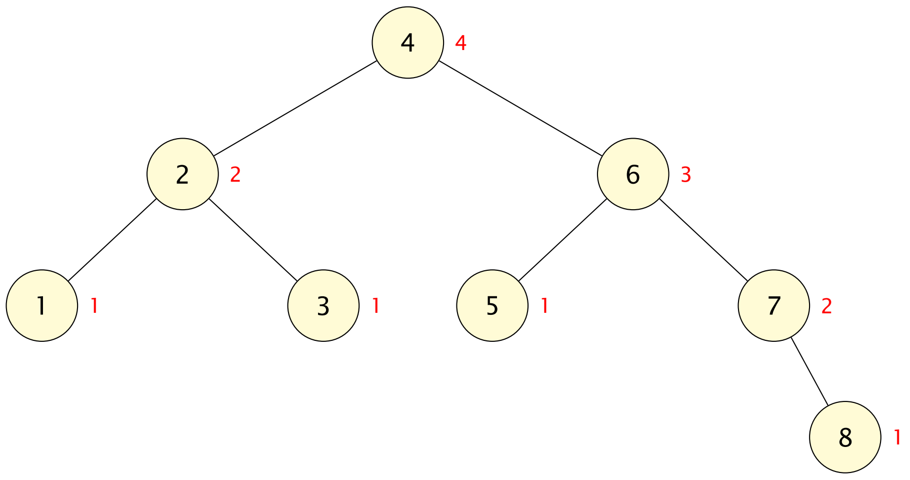
exsAVLInsDelQ1a07



Answer 22 Insert Lists and Delete Items

```
exsAVLInsDelQ1a08 \
 = insertListAVLT(listQ1a[:8], EmptyABT())
```

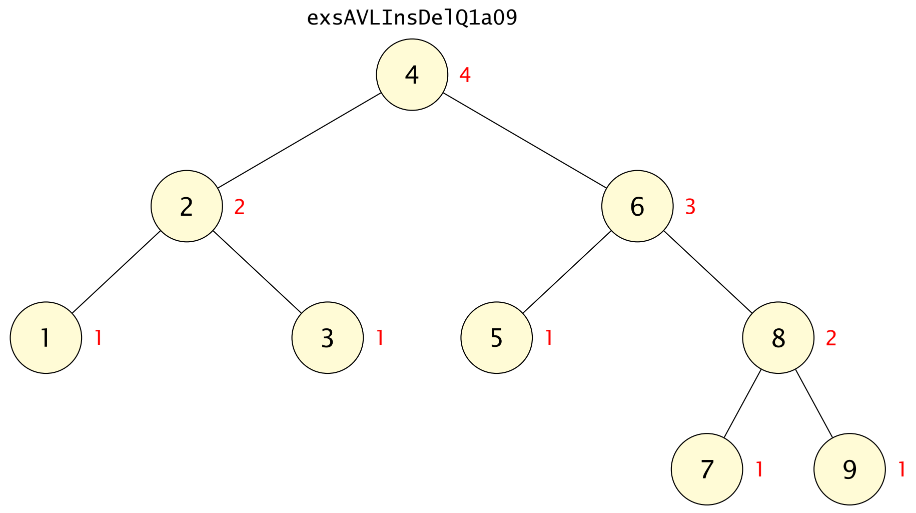
exsAVLInsDelQ1a08



Answer 22 Insert Lists and Delete Items

```
exsAVLInsDelQ1a09 \
 = insertListAVLT(listQ1a[:9], EmptyABT())
```

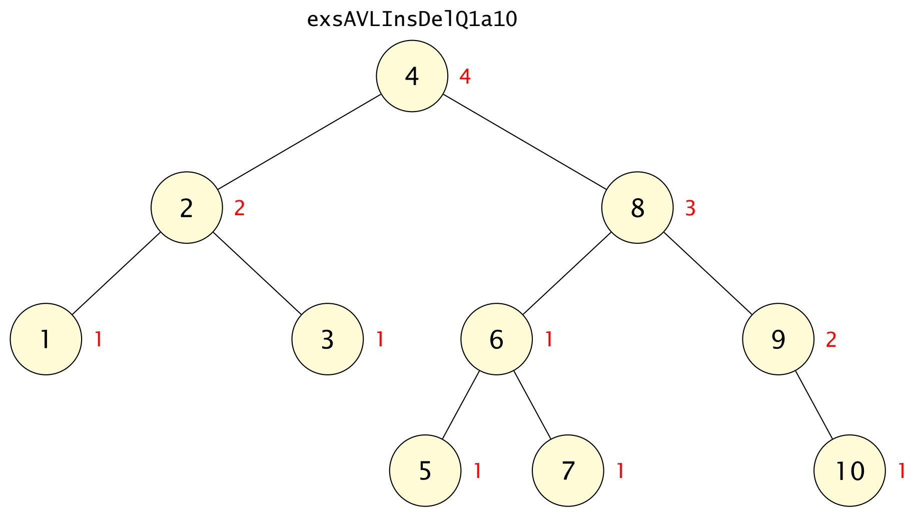
Left rotation about 7



Answer 22 Insert Lists and Delete Items

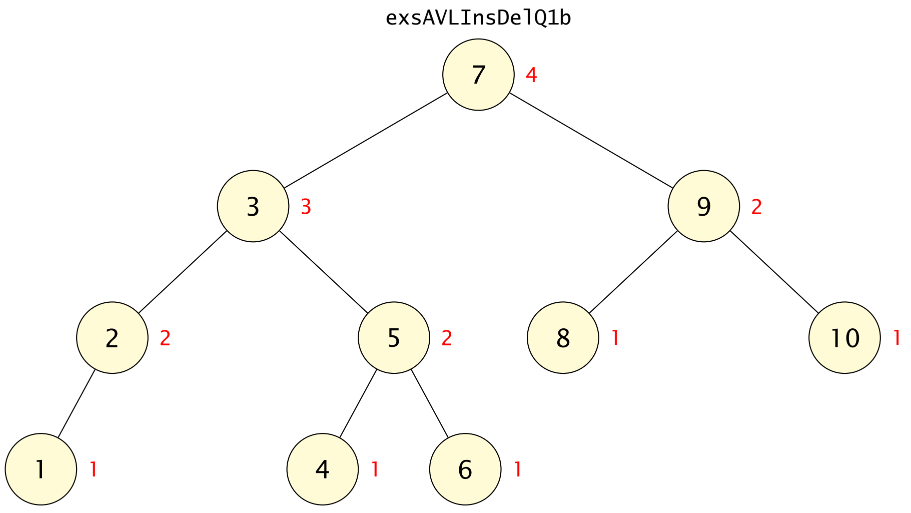
```
exsAVLInsDelQ1a10 \
 = insertListAVLT(listQ1a[:10], EmptyABT())
```

Left rotation around 6



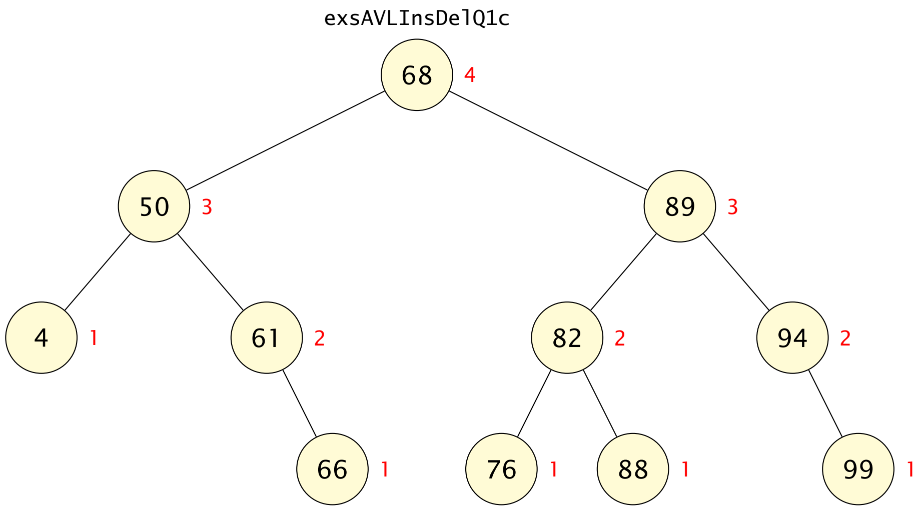
Answer 22 Insert Lists and Delete Items

```
listQ1b = [10,9,8,7,6,5,4,3,2,1]
exsAVLInsDelQ1b = insertListAVLT(listQ1b, EmptyABT())
```



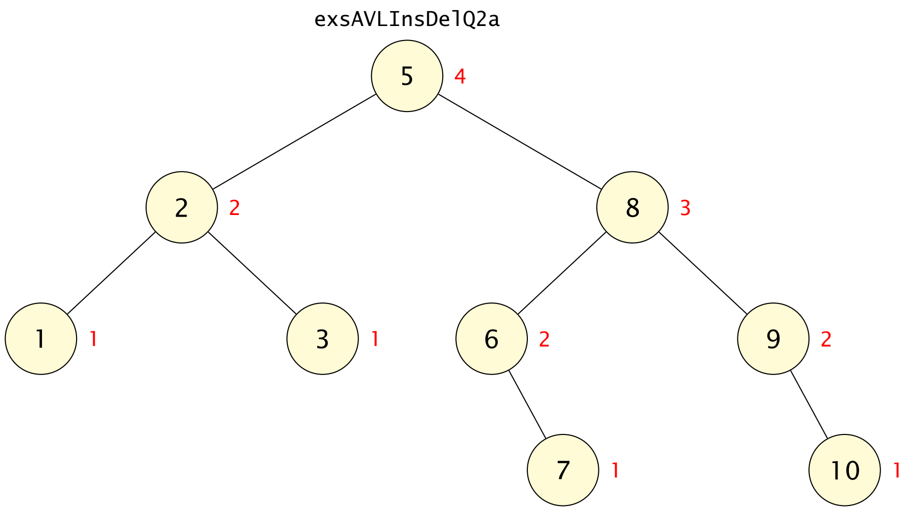
Answer 22 Insert Lists and Delete Items

```
listQ1c = [68,88,61,89,94,50,4,76,66,82,99]
exsAVLInsDelQ1b = insertListAVLT(listQ1c, EmptyABT())
```



Answer 22 Q2(a) Delete 4th Item

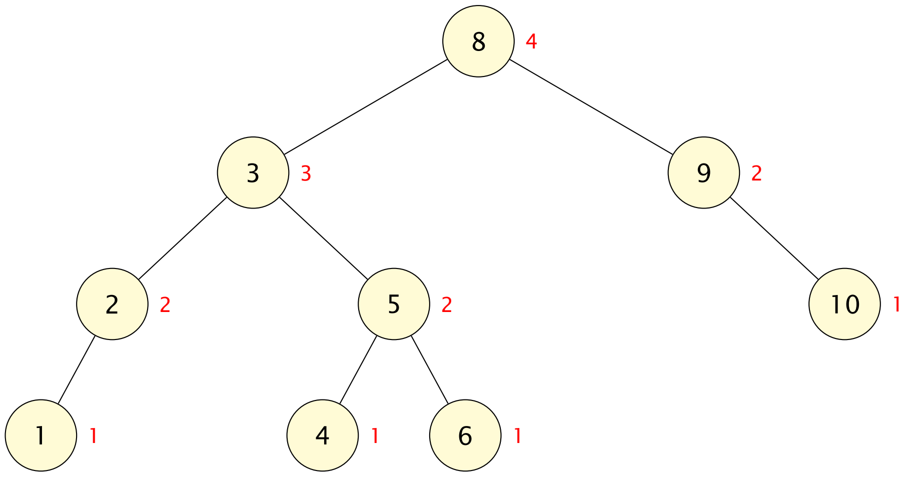
```
listQ1a = [1,2,3,4,5,6,7,8,9,10]
exsAVLInsDelQ2a \
 = deleteAVLT(listQ1a[3], exsAVLInsDelQ1a)
```



Answer 22 Q2(b) Delete 4th Item

```
listQ1b = [10,9,8,7,6,5,4,3,2,1]
exsAVLInsDelQ2b \
 = deleteAVLT(listQ1b[3], exsAVLInsDelQ1b)
```

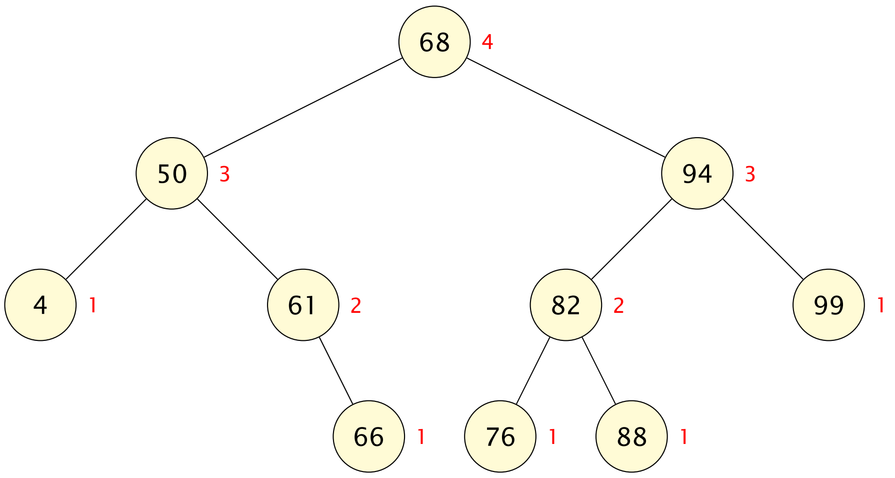
exsAVLInsDelQ2b



Answer 22 Insert Lists and Delete Items

```
listQ1c = [68,88,61,89,94,50,4,76,66,82,99]
exsAVLInsDelQ2c \
 = deleteAVLT(listQ1c[3], exsAVLInsDelQ1c)
```

exsAVLInsDelQ2c



Go to Activity

Activity 23 Deleting Inserted List

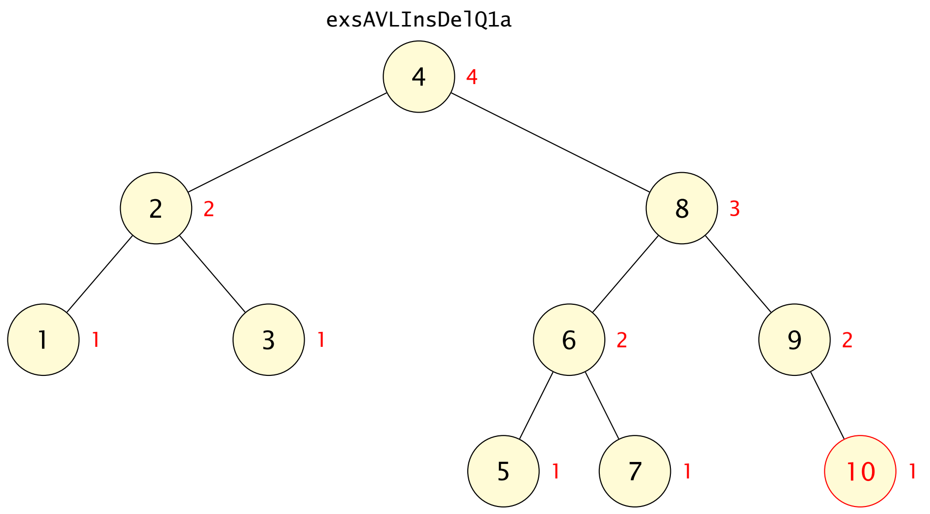
- Using listQ1a show that deleting the elements of the list from the tree one by one in reverse order does not result in the reverse sequence of AVL trees

```
listQ1a = [1,2,3,4,5,6,7,8,9,10]
exsAVLInsDelQ1a = insertListAVLT(listQ1a, EmptyABT())
```

Go to Answer

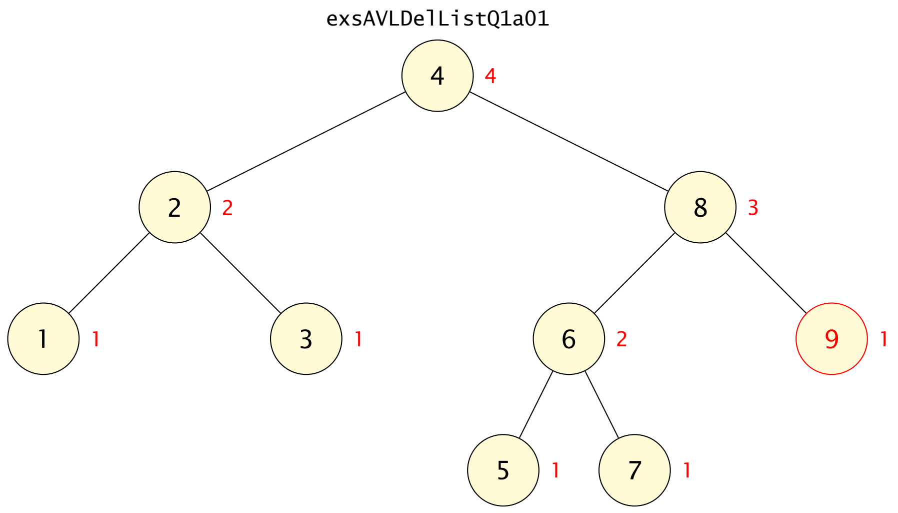
Answer 23 Deleting Inserted List

```
listQ1a = [1,2,3,4,5,6,7,8,9,10]
exsAVLInsDelQ1a = insertListAVLT(listQ1a, EmptyABT())
# delete listQ1a[-1]
```



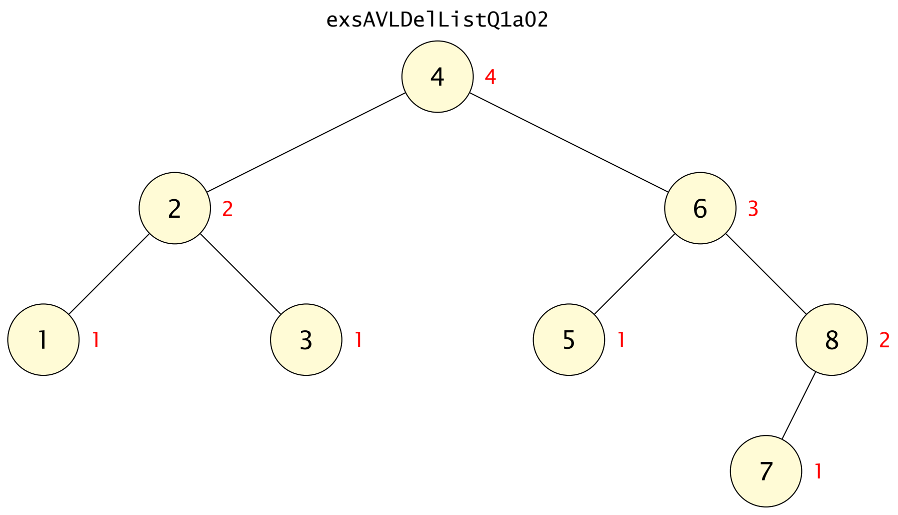
Answer 23 Deleting Inserted List

```
exsAVLDelListQ1a01 \
 = deleteAVLT(listQ1a[-1], exsAVLInsDelQ1a)
# delete listQ1a[-2]
```



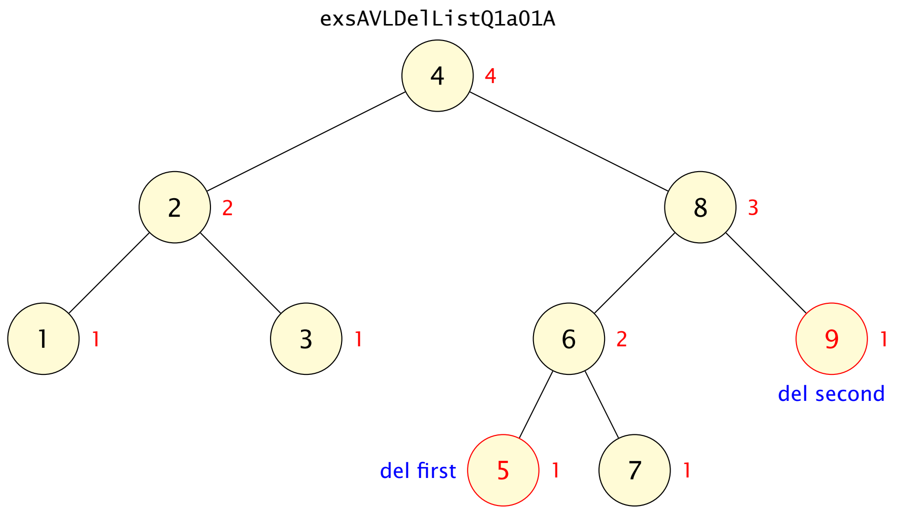
Answer 23 Deleting Inserted List

```
exsAVLDelListQ1a02 \
 = deleteAVLT(listQ1a[-2], exsAVLDelListQ1a01)
# Right rotation about node 8
```



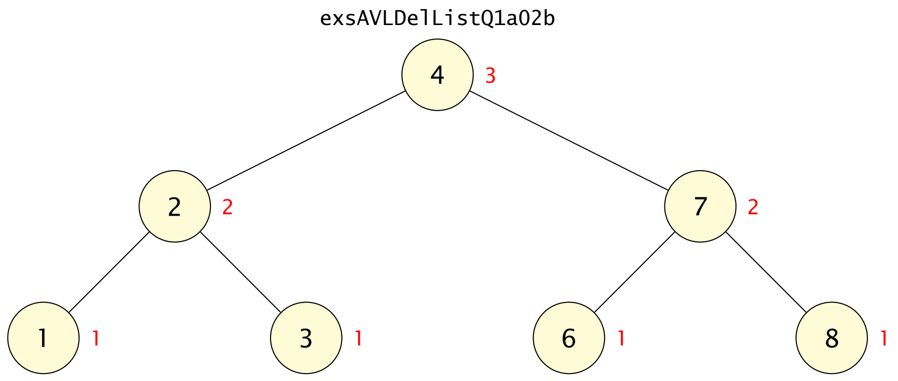
Answer 23 Deleting Inserted List

```
exsAVLDelListQ1a01 \
 = deleteAVLT(listQ1a[-1], exsAVLInsDelQ1a)
# delete 5 first followed by 9
```



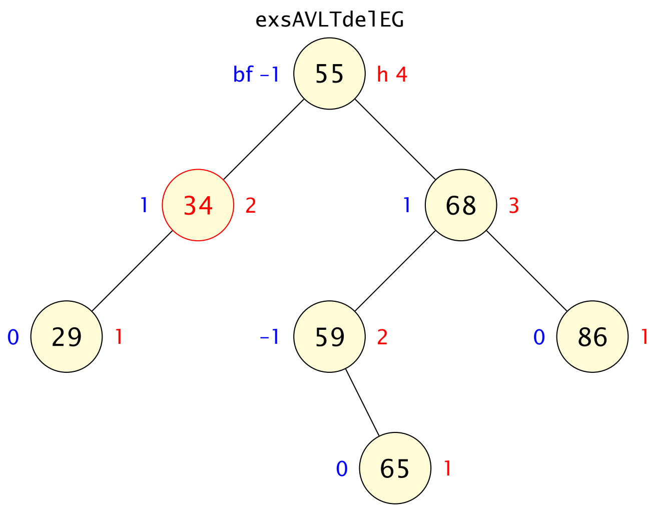
Answer 23 Deleting Inserted List

```
exsAVLDelListQ1a02a \
 = deleteAVLT(listQ1a[-6], exsAVLDelListQ1a01)
exsAVLDelListQ1a02b \
 = deleteAVLT(listQ1a[-2], exsAVLDelListQ1a02a)
# Double rotation: left about 6, right about 8
```



Activity 24 Delete with Rebalance

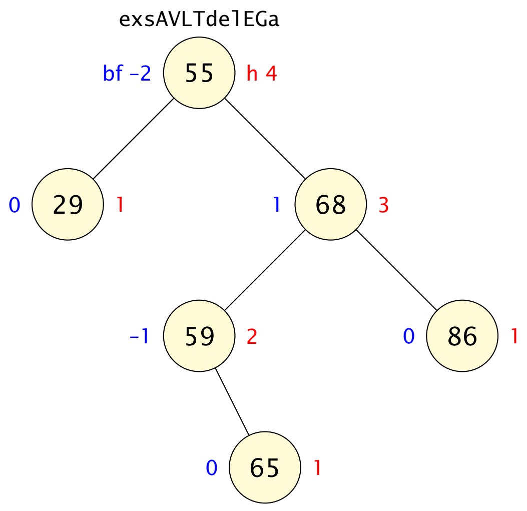
- Example from Specimen Exam (2016) Q 8
- Redraw the tree with node 34 deleted and tree rebalanced. Note here we have height of empty tree as 0 and singleton node as 1



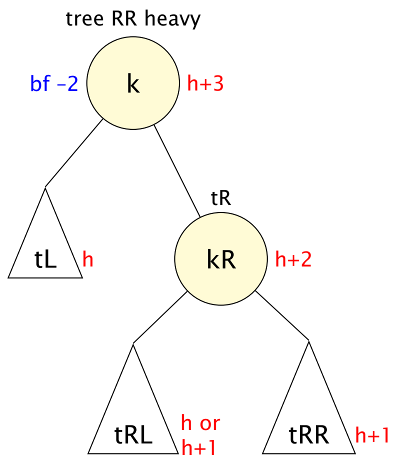
Go to Answer

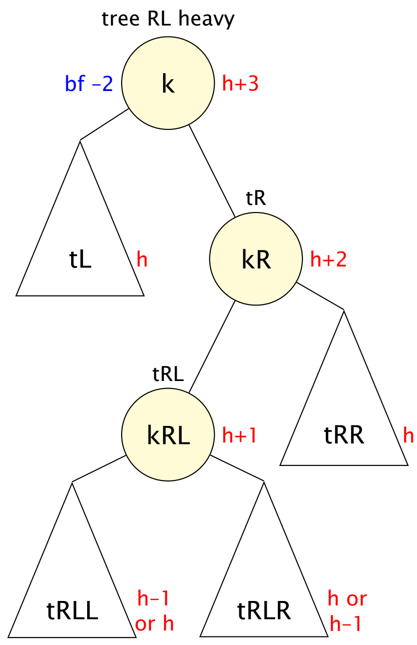
Answer 24 Delete with Rebalance

- Here is the tree with node 34 deleted but not rebalanced
- The new balance factor for the root is -2 so two possible transformations — RR heavy or RL heavy



Go to Activity





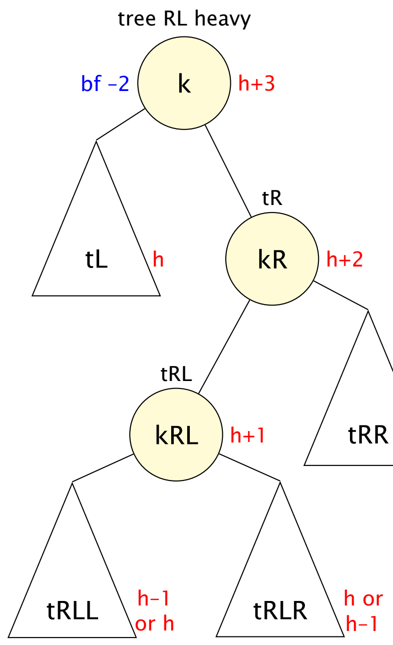
Go to Activity

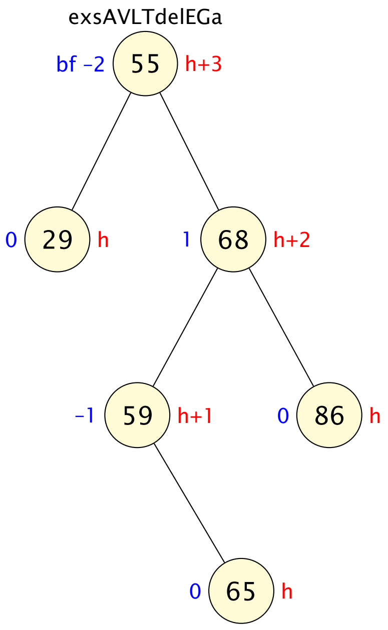
- Exercise: Identify the parts of the tree given in the question with the names given for key nodes and subtrees given in the above diagrams
- Which of the two cases is the given tree an instance of ?

Go to Activity

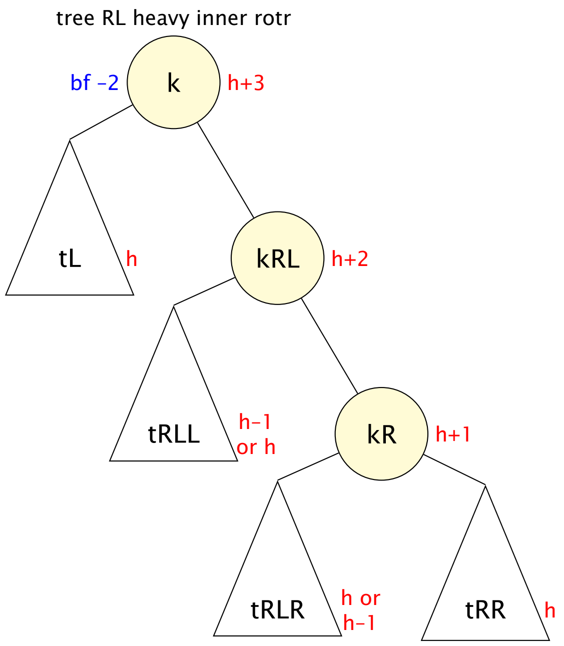
- Key k is 55
- Key kR is 68
- Key kRL is 59
- Subtree tL is rooted at 29
- Subtree tRL is rooted at 59
- Subtree RR is rooted at 86
- Subtree tRLL is an empty tree
- Subtree tRLR is rooted at 65
- The given tree is an instance of RL heavy
- This requires a double rotation to rebalance

Go to Activity

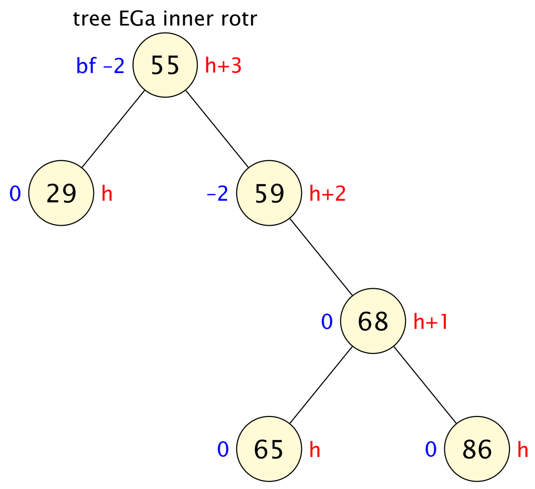




Go to Activity



Go to Activity



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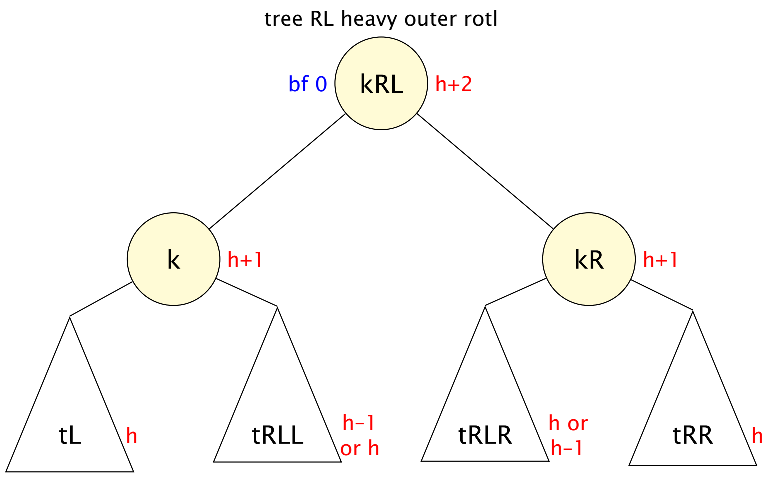

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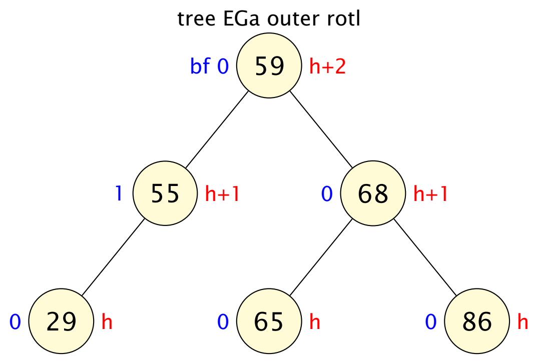
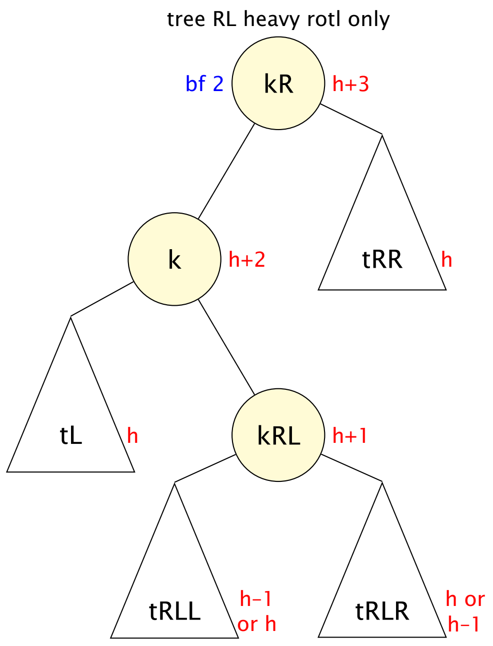


Go to Activity

- Exercise: what would have happened if we had chosen only to do a left rotation around the root ?

Go to Activity



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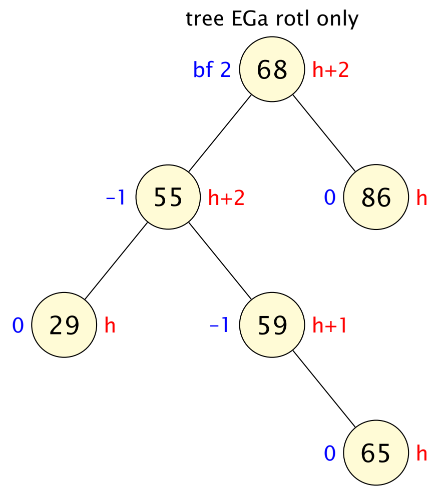


- This tree is LR heavy and could be rebalanced via a further double rotation but obviously this would be extra work compared to getting the correct double rotation in the first place

Go to Activity

- **Key point** when performing a rebalance, check which case applies
- **LL heavy** right rotation
- **LR heavy** inner left rotation, right outer rotation
- **RL heavy** inner right rotation, left outer rotation
- **RR heavy** left rotation
- See the notes for the details

Go to Activity

ToC

6.6 AVL Tree Performance

- While a height balanced tree may not always have the minimum possible height, it has the advantage that it will always be reasonably small
- For a tree with n items we shall show that the maximum number of steps to insert, delete or retrieve an item is $O(\log n)$
- Finding the maximum height of a tree with n items is equivalent to finding the minimum number of items, T_h in a tree of height h
- For $h = 0$ we have an empty tree so $T_0 = 0$
- For T_1 we have a singleton item so $T_1 = 1$
- In general for $h \geqslant 2$ we have $T_h = 1 + T_{h-1} + T_{h-2}$

  since the tree must be balanced and each subtree must have a minimum number of items
- The sequence T_h looks very similar to the Fibonacci sequence
- $F_0 = 0,\ F_1 = 1$
- $F_k = F_{k-1} + F_{k-2},\ k \geqslant 2$

- The Fibonacci sequence appeared in a work by Leonardo Fibonacci Pisano, who also popularized the Hindu-Arabic numeral system via his 1202 book *Liber Abaci (Book of Calculations)*.
- The sequence also appeared in Indian mathematics much earlier.
- The Fibonacci numbers have lots of interesting properties and turn up in many places in nature
- In our case we have $T_h = F_{h+2} - 1$
- Deriving $T_h = F_{h+2} - 1$
- Let $R_h = T_h - T_{h-1} = 1 + T_{h-2}$
- Then $R_{h+2} = 1 + T_h = 1 + 1 + T_{h-1} + T_{h-2} = R_{d+1} + R_d$
- $R_2 = 1 + T_0 = 1 + 0 = 1 = F_2$ and

  $R_3 = 1 + T_1 = 1 + 1 = 2 = F_3$
- Hence $R_h = F_h, \ \forall h \geqslant 2$
- Hence $T_h = F_{h+2} - 1, \ \forall h \geqslant 0$
- Miller & Ranum approach
- **Level** number of edges from root to node
- **Height** maximum level of any node in the tree — this is one less than my definition
- N_h is the minimum number of nodes in an AVL tree of height h
- $N_0 = 1$ since tree of one node has no edges

  $N_1 = 2$
- $N_h = 1 + N_{h-1} + N_{h-2}, \ h \geqslant 2$
- Let $S_h = N_h - N_{h-1} = 1 + N_{h-2}$
- Then $S_{h+2} = 1 + N_h = 1 + 1 + N_{h-1} + N_{h-2} = S_{d+1} + S_d$
- $S_2 = 1 + N_0 = 1 + 1 = 2 = F_3$ and

  $S_3 = 1 + N_1 = 1 + 2 = 3 = F_4$
- Hence $S_h = F_{h+1}, \ \forall h \geqslant 2$
- Hence $N_h = F_{h+3} - 1, \ \forall h \geqslant 0$
- $P(h) : T_h = F_{h+2} - 1$ proof by induction
- **Basis** $P(0), P(1)$
- $T_0 = 1$ and $F_2 - 1 = 1 - 1 = 0$
- $T_1 = 1$ and $F_3 - 1 = 2 - 1 = 1$
- **Inductive step** $\forall k \, P(k) \Rightarrow P(k+1)$
- $T_k = 1 + T_{k-1} + T_{k-2}$

  $= 1 + (F_{k+1} - 1) + (F_k - 1)$

  $= F_{k+2} - 1$

- Hence $T_h = F_{h+2} - 1, \ \forall h \geqslant 0$
- There is a closed-form solution for the Fibonacci sequence known as the Euler-Binet Formula (see also A formula for Fib(n))
- $F_k = \dfrac{\phi^k - (1-\phi)^k}{\sqrt{5}}$
- ϕ is the Golden mean
- $\phi = \dfrac{1}{\phi - 1} = \dfrac{1+\sqrt{5}}{2} \approx 1.61803\ldots$
- Hence $T_h = \dfrac{\phi^{h+2} - (1-\phi)^{h+2}}{\sqrt{5}} - 1$
- Since $(1-\phi) < 1$ then for large h we have
- $T_h = n \approx \dfrac{\phi^{h+2}}{\sqrt{5}} - 1 \rightarrow \log(\sqrt{5}(n+1)) \approx (h+2)\log\phi$
- Hence in the worst case, the height of a AVL tree is $O(\log n)$

6.6.1 Proof of Euler-Binet Formula

- Proof of the Euler-Binet Formula is not required for M269 but here is a brief summary
- **Proof by Induction**
- Let $P(n) = F_n = \dfrac{\phi^n - (1-\phi)^n}{\sqrt{5}}$
- **Basis for Induction**
- $P(0)$ is true since

  $$\frac{\phi^0 - (1-\phi)^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 == F_0$$
- $P(1)$ is true since

  $$\frac{\phi^1 - (1-\phi)^1}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right) - \left(1 - \left(\frac{1+\sqrt{5}}{2}\right)\right)}{\sqrt{5}}$$
- $= 1 == F_1$
- **Induction Hypothesis Step**
- Show $P(j) : 0 \leqslant j \leqslant k+1 \Rightarrow P(k+2)$
- $\phi^{k+2} - (1-\phi)^{k+2} = \phi^2\phi^k - (1-\phi)^2(1-\phi)^k$ and
- $\phi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1}{4}(1 + 2\sqrt{5} + 5) = 1 + \phi$
- $(1-\phi)^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2 = 1 + (1-\phi)$ hence
- $\phi^{k+2} - (1-\phi)^{k+2} = (1+\phi)\phi^k - (1 + (1-\phi))(1-\phi)^k$
- $= \left(\phi^k - (1-\phi)^k\right) + \left(\phi^{k+1} - (1-\phi)^{k+1}\right)$
- $= \sqrt{5}(F_k + F_{k+1})$ by inductive hypothesis

- $= \sqrt{5} F_{k+2}$ by Fibonacci definition
- Hence $\forall\, n \in \mathbb{N} : F_n = \dfrac{\phi^n - (1-\phi)^n}{\sqrt{5}}$
- The above proof confirms the formula but here is a derivation
- Define $T(x, y) = (y, x + y)$
- Then $T^n(0, 1) = (F_n, F_{n+1})$ (proof by induction)
- Now find λ_1, λ_2 and $(x_1, y_1), (x_2, y_2)$

  so $T(x_1, y_1) = \lambda_1(x_1, y_1)$ and $T(x_2, y_2) = \lambda_2(x_2, y_2)$

  and $(0, 1) = p_1(x_1, y_1) + p_2(x_2, y_2)$
- $T(x, y) = (y, x + y) = \lambda(x, y)$

  $\rightarrow x + y = \lambda x$ and $x = \lambda y \rightarrow \lambda^2 - \lambda - 1 = 0$

  $\rightarrow \lambda_1 = \phi$ and $\lambda_2 = 1 - \phi$

  and $T(1, \phi) = (\phi, 1 + \phi) = \phi(1, \phi)$

  $T(1, 1 - \phi) = (1 - \phi, 1 + (1 - \phi)) = (1 - \phi)(1, 1 - \phi)$
- $(0, 1) = \frac{1}{\sqrt{5}}(1, \phi) - \frac{1}{\sqrt{5}}(1, 1 - \phi)$ confirm by inspection
- $(F_n, F_{n+1}) = T^n(0, 1)$

  $= \frac{1}{\sqrt{5}} T^n(1, \phi) - \frac{1}{\sqrt{5}} T^n(1, 1 - \phi)$

  $= \frac{1}{\sqrt{5}} \phi^n(1, \phi) - \frac{1}{\sqrt{5}}(1 - \phi)^n(1, 1 - \phi)$

  $= \frac{1}{\sqrt{5}}(\phi^n, \phi^{n+1}) - \frac{1}{\sqrt{5}}((1 - \phi)^n, (1 - \phi)^{n+1})$
- Hence $F_n = \dfrac{\phi^n - (1-\phi)^n}{\sqrt{5}}$

ToC

7 AVL Tree Application — Sets

- Ordered sets and ordered maps are important data types in programming
- Some programming languages have them as builtin types (Python) or supply them as standard libraries (C++, C#, Java, Scala, Haskell, ML)
- This section describes an example implementation based on Blelloch et al. (2016) and Adams (1993)
- Note that this example also shows the use of recursive thinking in practice
- In Python the documentation for Sets is at Set Types and for Dictionaries (Maps) at Mapping Types — dict
- The Python implementation can be found at the Python Developer's Guide and the source code for Sets is at setobject.c — the implementation is in C using hash tables — see How is set() implemented?

- In Haskell the documentation and implementation of Sets is at Containers: Data.Set and for Maps at Containers: Data.Map.Strict — both of these are from the package containers: Assorted concrete container types
- The Haskell implementation uses size balanced trees — this is similar to AVL balanced trees
- For an introduction see containers - Introduction and Tutorial
- For an overview of Sets see Containers: Sets
- M269 Unit 5 has representation of graphs for various algorithms — here are some references for that topic for future notes
- For Haskell graph libraries see
  - fgl: Martin Erwig's Functional Graph Library
  - graphs: A simple monadic graph library by Edward Kmett
  - Data.Graph based on King and Launchbury (1995)

7.1 Set Representation

7.1.1 Split, Join

- The representation of sets uses the ABTree data type but with generalised versions of the split and join functions
- While implementing sets in AVL trees is not directly part of M269, it gives good examples of recursive thinking in an important application
- Note that the data item is used as the key for a node in the tree — in practice there would be separate key and data
- splitAVLS takes a key k and an AVL tree t and returns two trees tL and tR and a boolean b — tL and tR have elements less than and greater than k respectively and b indicates if k was in t
- splitLastAVLS, splitFirstAVLS take an AVL tree t and return the largest, smallest elemeent k respectively and the rest of the tree — splitFirstAVLS is similar to splitAVLT at line 926 on page 73
- join2AVLS, joinAVLS take two AVL trees, tL, tR where all elements of tL are less than all elements of tR and returns a new AVL tree — joinAVLS also takes a key k with a value in between the elements of the two trees — join2AVLS is similar to joinAVLT at line 917 on page 73
- exposeABT takes apart an augmented tree, t to give (tL, k, tR)
- Here is a reminder of some of the ABTree constructors and inspectors from file M269TutorialBinaryTrees2022.py

```
def mkEmptyABT() -> ABTree :
  return EmptyABT()

def mkNodeABT(x: T,t1: ABTree,t2: ABTree) -> ABTree :
  h = 1 + max(getHeightABT(t1),getHeightABT(t2))
  return NodeABT(h,x,t1,t2)

def isEmptyABT(t: ABTree) -> bool :
  return t == EmptyABT()
```

- And here is the additional inspector `expose` in `M269TutorialBinaryTrees2022AVLSets.py`

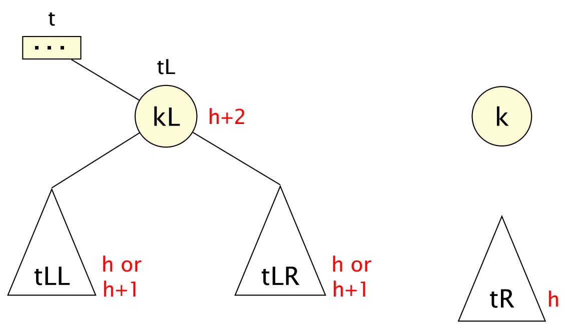
```
11 def exposeABT(t: ABTree) -> (ABTree, T, ABTree) :
12   if isEmptyABT(t) :
13     raise RuntimeError("exposeABT␣applied␣to␣EmptyABT()")
14   else :
15     tL = getLeftABT(t)
16     k  = getDataABT(t)
17     tR = getRightABT(t)
18     return (tL,k,tR)
```

- `joinAVLS` take a key, `k`, two AVL trees, `tL`, `tR` where all elements of `tL` are less than `k` which is less than all elements of `tR` and returns a new AVL tree

```
20 def joinAVLS(k: T, tL: ABTree, tR: ABTree) -> ABTree :
21   if getHeightABT(tL) > getHeightABT(tR) + 1 :
22     return joinRightAVLS(k,tL,tR)
23   elif getHeightABT(tR) > getHeightABT(tL) + 1 :
24     return joinLeftAVLS(k,tL,tR)
25   else :
26     return mkNodeABT(k,tL,tR)
```

- `joinRightAVLS` description is in the following diagrams

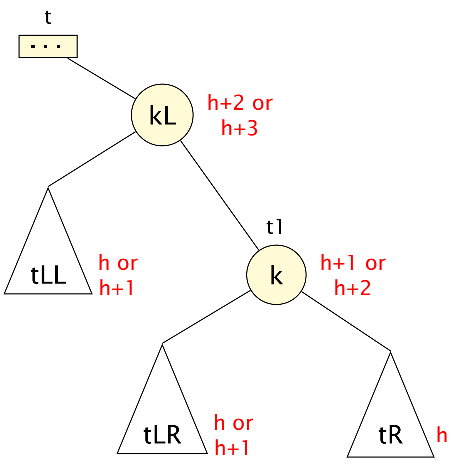
```
28 def joinRightAVLS(k: T, tL: ABTree, tR: ABTree) -> ABTree :
29   (tLL,kL,tLR) = exposeABT(tL)
30   if getHeightABT(tLR) <= getHeightABT(tR) + 1 :
31     t1 = mkNodeABT(k,tLR,tR)
32     if getHeightABT(t1) <= getHeightABT(tLL) + 1 :
33       return mkNodeABT(kL,tLL,t1)
34     else :
35       return rotl(mkNodeABT(kL,tLL,(rotr(t1))))
36   else :
37     t2 = joinRightAVLS(k,tLR,tR)
38     t3 = mkNodeABT(kL,tLL,t2)
39     if getHeightABT(t2) <= getHeightABT(tLL) + 1 :
40       return t3
41     else :
42       return rotl(t3)
```



- The base case (line 30 on page 91) of `joinRightAVLS` follows the right spine of `t` to a node `kL` for which

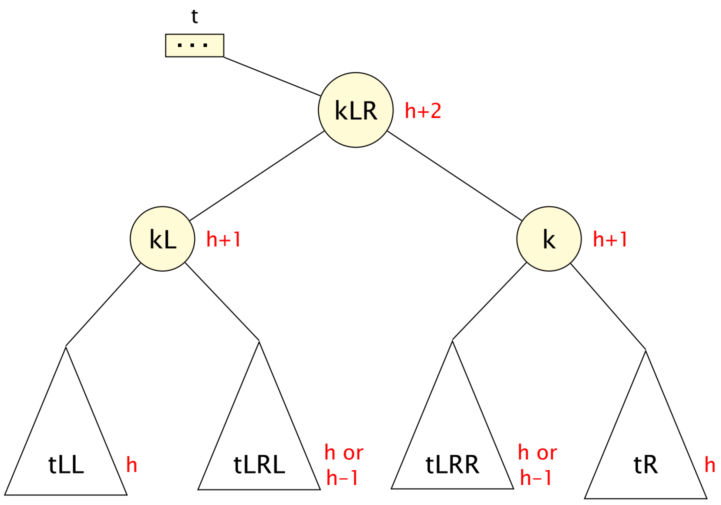
```
getHeightABT(tL) > getHeightABT(tR) + 1
getHeightABT(tLR) <= getHeightABT(tR) + 1
```

- We then connect `tL`, `k` and `tR`



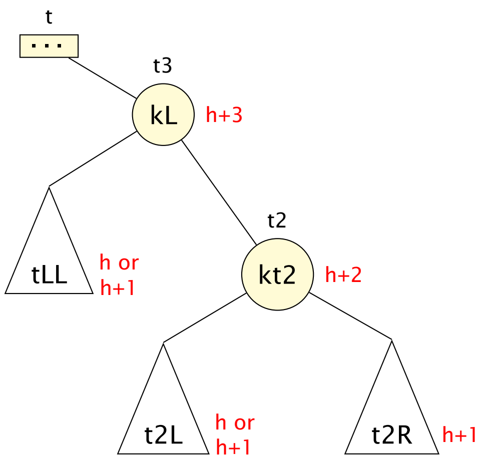
- Needs double rotation if

```
getHeightABT(t1) > getHeightABT(tLL) + 1
```



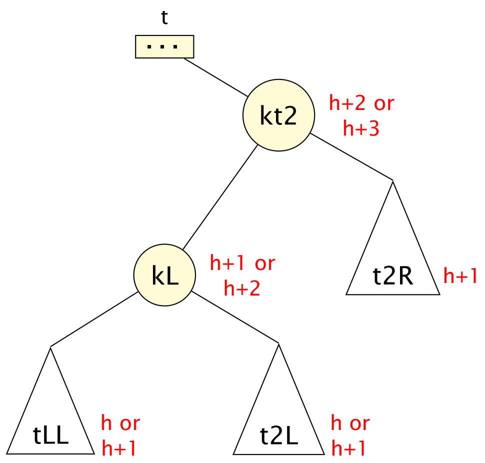
- The recursive case (line 36 on page 91) of `joinRightAVLS` follows the right spine further

```
getHeightABT(tLR) > getHeightABT(tR) + 1
```



- A single left rotation is needed if

```
getHeightABT(t2) > getHeightABT(tLL) + 1
```



```python
def joinLeftAVLS(k: T, tL: ABTree, tR: ABTree) -> ABTree :
  (tRL, kR, tRR) = exposeABT(tR)
  if getHeightABT(tRL) <= getHeightABT(tL) + 1 :
    t1 = mkNodeABT(k,tL,tRL)
    if getHeightABT(t1) <= getHeightABT(tRR) + 1 :
      return mkNodeABT(kR,t1,tRR)
    else :
      return rotr(mkNodeABT(kR,(rotl(t1)),tRR))
  else :
    t2 = joinLeftAVLS(k,tL,tRL)
    t3 = mkNodeABT(kR,t2,tRR)
    if getHeightABT(t2) <= getHeightABT(tRR) + 1 :
      return t3
    else :
      return rotr(t3)
```

Activity 25 joinLeftAVLS Diagrams

- joinLeftAVLS is the mirror image of joinRightAVLS
- Produce the equivalent diagrams describing the function

Go to Answer

Answer 25 joinLeftAVLS Diagrams

- TODO: Answer 25 joinLeftAVLS Diagrams

Go to Activity

Activity 26 joinLeftAVLS Bug

- A previous version of joinLeftAVLS had a bug (beware copy/paste) — see below
- What would happen if the elements of [10,9,8,7,6] were given as input ?

```python
def joinLeftAVLS(k: T, tL: ABTree, tR: ABTree) -> ABTree :
  (tRL, kR, tRR) = exposeABT(tR)
  if getHeightABT(tRL) <= getHeightABT(tL) + 1 :
    t1 = mkNodeABT(k,tL,tRL)
    if getHeightABT(t1) <= getHeightABT(tRR) + 1 :
      return mkNodeABT(kR,t1,tRR)
    else :
      return rotr(mkNodeABT(kR,(rotl(t1)),tRR))
  else :
    t2 = joinRightAVLS(k,tL,tRL)
    t3 = mkNodeABT(kR,t2,tRR)
    if getHeightABT(t2) <= getHeightABT(tRR) + 1 :
      return t3
    else :
      return rotr(t3)
```

Go to Answer

Answer 26 joinLeftAVLS Bug

- TODO: Answer 26 joinLeftAVLS Bug

Go to Activity

```
def splitLastAVLS(t: ABTree) -> (ABTree,T) :
  (tL,k,tR) = exposeABT(t)
  if isEmptyABT(tR) :
    return (tL,k)
  else :
    (tR1,k1) = splitLastAVLS(tR)
    return (joinAVLS(k,tL,tR1),k1)

def splitFirstAVLS(t: ABTree) -> (ABTree,T) :
  (tL, k, tR) = exposeABT(t)
  if isEmptyABT(tL) :
    return (tR,k)
  else :
    (tL1,k1) = splitFirstAVLS(tL)
    return (joinAVLS(k,tL1,tR),k1)
```

```
def splitAVLS(k: T,t: ABTree) -> (ABTree,bool,ABTree) :
  if isEmptyABT(t) :
    return (mkEmptyABT(),False,mkEmptyABT())
  else :
    (tL, k1, tR) = exposeABT(t)
    if k == k1 :
      return (tL,True,tR)
    elif k < k1 :
      (tLL, b, tLR) = splitAVLS(k,tL)
      return (tLL, b, (joinAVLS(k1,tLR,tR)))
    else :
      (tRL, b, tRR) = splitAVLS(k,tR)
      return ((joinAVLS(k1,tL,tRL)), b, tRR)
```

```
def join2AVLS(tL: ABTree,tR: ABTree) -> ABTree :
  if isEmptyABT(tL) :
    return tR
  else :
    (tL1, k) = splitLastAVLS(tL)
    return joinAVLS(k,tL1,tR)
```

ToC

7.2 Set Operations

- **Set Operations**
- insertAVLS(t,k) inserts a key, k, into a tree, t
- deleteAVLS(t,k) deletes key, k, from a tree, t, if it is in the tree
- unionAVLS(t1,t2) takes two AVL trees whose values may overlap, and returns the union as a tree
- intersectAVLS(t1,t2) takes two AVL trees and returns the intersection as a tree
- disjoint(t1,t2) takes two AVL trees and returns True if and only if they have no members in common
- differenceAVLS t1 t2 takes two AVL trees and returns the elements that are in t1 but not t2
- subsetAVLS(t1,t2) takes two AVL trees and returns True if and only if every member of t1 is a member of t2

```
def insertAVLS(t: ABTree,k: T) -> ABTree :
  (tL, found, tR) = splitAVLS(k,t)
  return joinAVLS(k,tL,tR)

def deleteAVLS(t: ABTree,k: T) -> ABTree :
  (tL, found, tR) = splitAVLS(k,t)
  return join2AVLS(tL,tR)

def insertListAVLS(t: ABTree,xs: [T]) -> ABTree :
  if xs == [] :
    return t
  else :
    return insertListAVLS(insertAVLS(t,xs[0]),xs[1:])

def setFromListAVLS(xs: [T])-> ABTree :
  return insertListAVLS(mkEmptyABT(),xs)
```

```
def unionAVLS(t1: ABTree,t2: ABTree) -> ABTree :
  if   isEmptyABT(t1) :
    return t2
  elif isEmptyABT(t2) :
    return t1
  else :
    (t2L, k2, t2R) = exposeABT(t2)
    (t1L, found, t1R) = splitAVLS(k2,t1)
    tL = unionAVLS(t1L,t2L)
    tR = unionAVLS(t1R,t2R)
    return joinAVLS(k2,tL,tR)
```

- `unionAVLS(t1,t2)` returns the set of all members of `t1` or `t2` (or both)

```
def intersectAVLS(t1: ABTree,t2: ABTree) -> ABTree :
  if   isEmptyABT(t1) :
    return mkEmptyABT()
  elif isEmptyABT(t2) :
    return mkEmptyABT()
  else :
    (t2L, k2, t2R) = exposeABT(t2)
    (t1L, found, t1R) = splitAVLS(k2,t1)
    tL = intersectAVLS(t1L,t2L)
    tR = intersectAVLS(t1R,t2R)
    if found :
      return joinAVLS(k2,tL,tR)
    else :
      return join2AVLS(tL,tR)
```

- `intersectAVLS(t1,t2)` returns the set of all members of both `t1` and `t2`
- Notice it needs the `if` statement to check that a member of `t2` is a member of `t1`

```
def disjointAVLS(t1: ABTree,t2: ABTree) -> ABTree :
  if   isEmptyABT(t1) :
    return True
  elif isEmptyABT(t2) :
    return True
  else :
    (t2L, k2, t2R) = exposeABT(t2)
    (t1L, found, t1R) = splitAVLS(k2,t1)
    return (not found
            and disjointAVLS(t1L,t2L)
            and disjointAVLS(t1R,t2R))
```

- `disjoint(t1,t2)` returns `True` if there are no elements in common
- If an element in common is found then `False` is returned
- Note that the behaviour of `splitAVLS()` ensures the search space is reduced at each recursive call

```
def differenceAVLS(t1: ABTree,t2: ABTree) -> ABTree :
  if   isEmptyABT(t1) :
    return mkEmptyABT()
  elif isEmptyABT(t2) :
    return t1
  else :
    (t2L, k2, t2R) = exposeABT(t2)
    (t1L, found, t1R) = splitAVLS(k2,t1)
    tL = differenceAVLS(t1L,t2L)
    tR = differenceAVLS(t1R,t2R)
    return join2AVLS(tL,tR)
```

- differenceAVLS(t1,t2) returns the set of members of t1 that are not in t2
- On first reading it may be surprising there is no if statement
- Remember the behaviour of splitAVLS()

Activity 27 Set to Ascending List

- Write a function setToAscList which takes a set and returns the contents as an ascending list

Go to Answer

Answer 27 Set to Ascending List

- Probably the simplest solution at this stage is to use inOrderABT()

```
def setToAscList(t) :
  return inOrderABT(t)
```

```
Python3>>> list3 = [2,1,4,3,6,5,8,7,10,9]
Python3>>> t10 = setFromListAVLS(list3)
Python3>>> type(t10)
<class 'M269TutorialBinaryTrees2022.NodeABT'>
Python3>>> list4 = setToAscList(t10)
Python3>>> list4
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
Python3>>>
```

Answer 27 Set to Ascending List

- Of course, someone from the pure functional programming world would define a higher order function to capture the recursion pattern with setFoldr() (this is not part of M269)

```
def setFoldr(f,z,t) :
  def go(y,t) :
    if isEmptyABT(t) :
      return y
    else :
      (tL,x,tR) = exposeABT(t)
      return (go(f(x,(go(y,tR))),tL))
  return go(z,t)

def setToAscListA(t) :
  def cons(x,xs) :
    return ([x] + xs)
  return setFoldr(cons,[],t)
```

Go to Activity

Activity 28 Set Equality

- Write a function setEquality which takes two sets, t1 and t2 and returns True if they are equal and False otherwise

Go to Answer

Answer 28 Set Equality

- Answer 28 Set Equality

```
def setEquality(t1,t2) :
  list1 = setToAscList(t1)
  list2 = setToAscList(t2)
  return (len(list1) == len(list2)
          and list1 == list2)
```

```
Python3>>> list1
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
Python3>>> t1 = setFromListAVLS(list1)
Python3>>> list10
[2, 1, 4, 3, 6, 5, 8, 7, 10, 9]
Python3>>> t10 = setFromListAVLS(list10)
Python3>>> t1 == t10
False
Python3>>> setEquality(t1,t10)
True
Python3>>>
```

Go to Activity

Activity 29 Subset

- Write a function subsetAVLS that takes two sets t1, t2 and returns True if t1 is a subset of t2 and False otherwise
- t1 is a subset of t2 if every element of t1 is a member of t2

Go to Answer

Answer 29 Subset

```
165 def subsetAVLS(t1: ABTree,t2: ABTree) -> bool :
166   if   isEmptyABT(t1) :
167     return True
168   elif isEmptyABT(t2) :
169     return False
170   else :
171     (t1L, k1, t1R) = exposeABT(t1)
172     (t2L, found, t2R) = splitAVLS(k1,t2)
173     return (found
174             and subsetAVLS(t1L,t2L)
175             and subsetAVLS(t1R,t2R))
```

- How does this work ?
- The recursive case at line 170 checks that the key at the root of t1 is in t2 and recursively checks the sub-trees
- splitAVLS() ensures that the subtrees are the correct ones to be checked

Go to Activity

ToC

7.3 Sets — Implementation Points

- The only tree specific functions are joinAVLS, joinRightAVLS and joinLeftAVLS — AVL trees could be changed to size balanced or Red-Black trees with little to be changed

- The various sets operations use splitAVLS and joinAVLS or join2AVLS to avoid more complex algorithms — some implementations may inline the functions for efficiency
- The diagrams for joinRightAVLS are essential for the understanding of the base and recursive cases
- The unionAVLS, intersectAVLS and differenceAVLS functions are very similar in their usage of split and join
- From O'Sullivan et al. (2008, page 301) see Data Structures
- *Maps give us the same capabilities as hash tables do in other languages. Internally, a map is implemented as a balanced binary tree. Compared to a hash table, this is a much more efficient representation in a language with immutable data. This is the most visible example of how deeply pure functional programming affects how we write code: we choose data structures and algorithms that we can express cleanly and that perform efficiently, but our choices for specific tasks are often different [from] their counterparts in imperative languages.*
- See Curious about the HashTable performance issues

	Python		**Haskell**
Operation	**Average**	**Worst**	**Worst**
Member	$O(1)$	$O(n)$	$O(\log n)$
Union	$O(m+n)$		$O(m\log(\frac{n}{m}+1))$
Intersection	$O(\min(m,n))$	$O(m \times n)$	$O(m\log(\frac{n}{m}+1))$
Difference	$O(m)$		$O(m\log(\frac{n}{m}+1))$
Insert	$O(1)$	$O(n)$	$O(\log n)$
Delete	$O(1)$	$O(n)$	$O(\log n)$

- Python: Time Complexity
- Haskell: Data.Set and Data.Map.Strict
- Remember that actual behaviour will depend on the data and compiler settings

ToC

Commentary 5

5 Binary Tree Exercises

- Binary Tree shapes
- Generating Binary Trees
- Catalan Numbers (advanced)

ToC

8 Binary Tree Common Exercises

- This section contains some common exercises used in *Google Interview Questions*
- See the References section for Web sites with more examples
- Note that this section is not directly part of M269 and is here for interest and practice using recursion
- Further questions may be added to this section

8.1 Binary Tree Shapes

Activity 30 Shape Exercises

- `isSameShape(t1,t2)` takes two binary trees and returns `True` if they have the same shape
- `isMirrorShape(t1,t2)` takes two binary trees and returns `True` if they are a mirror of each other
- `isSymmetric(t)` takes a binary tree and returns `True` if it is symmetric
- `genMirrorShape(t)` takes a binary tree and returns the mirror of the tree

Go to Answer

Answer 30 Shape Exercises — `isSameShape(t1,t2)`

- `isSameShape(t1,t2)` takes two binary trees and returns `True` if they have the same shape

```
def isSameShape(t1: ABTree,t2: ABTree) -> bool :
  if isEmptyABT(t1) and isEmptyABT(t2) :
    return True
  elif isEmptyABT(t1) or isEmptyABT(t2) :
    return False
  else :
    (t1L,k1,t1R) = exposeABT(t1)
    (t2L,k2,t2R) = exposeABT(t2)
    return (isSameShape(t1L,t2L)
            and isSameShape(t1R,t2R))
```

Answer 30 Shape Exercises — `isMirrorShape(t1,t2)`

- `isMirrorShape(t1,t2)` takes two binary trees and returns `True` if they are a mirror of each other

```
def isMirrorShape(t1: ABTree,t2: ABTree) -> bool :
  if isEmptyABT(t1) and isEmptyABT(t2) :
    return True
  elif isEmptyABT(t1) or isEmptyABT(t2) :
    return False
  else :
    (t1L,k1,t1R) = exposeABT(t1)
    (t2L,k2,t2R) = exposeABT(t2)
    return (isMirrorShape(t1L,t2R)
            and isMirrorShape(t1R,t2L))
```

Answer 30 Shape Exercises — `isSymmetric(t)`

- `isSymmetric(t)` takes a binary tree and returns `True` if it is symmetric

```
def isSymmetric(t: ABTree) -> bool :
  return isMirrorShape(t,t)
```

Answer 30 Shape Exercises — genMirrorShape(t)

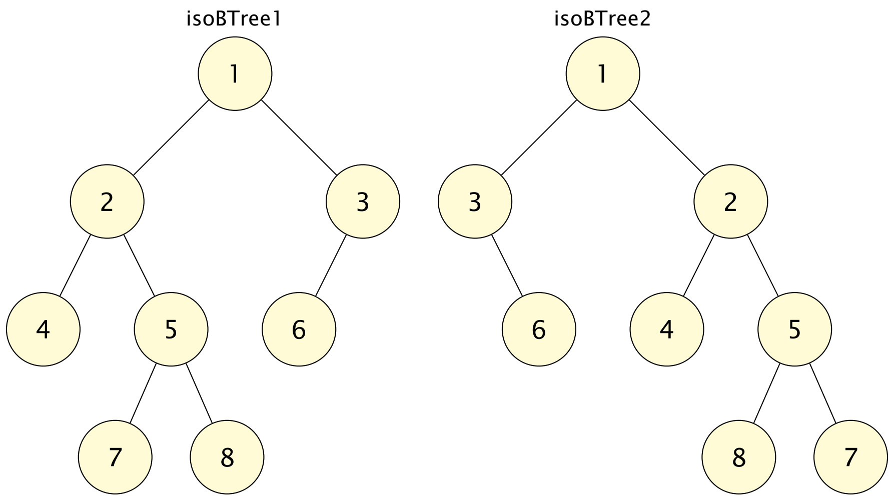
- genMirrorShape(t) takes a binary tree and returns the mirror of the tree

```
def genMirrorShape(t: ABTree) -> ABTree :
  if isEmptyABT(t) :
    return mkEmptyABT()
  else :
    (tL,x,tR) = exposeABT(t)
    return (mkNodeABT(x, genMirrorShape(tR),
                         genMirrorShape(tL)))
```

Go to Activity

8.1.1 Isomorphic Binary Trees

- Two binary trees are isomorphic if one can be obtained from the other by flipping the left and right subtrees. Two empty trees are isomorphic
- See Tree isomorphism problem
- See Is the recursive approach to binary tree isomorphism actually linear?



- isoBTree1, isoBTree2 are isomorphic with the following flips:

  (2,3), (EmptyBTree, 6), (7,8)

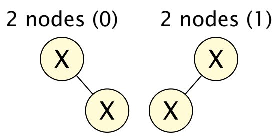
```
def isIsomorphic(t1 : ABTree, t2 : ABTree) -> bool :
  if isEmptyABT(t1) and isEmptyABT(t2) :
    return True
  elif isEmptyABT(t1) or isEmptyABT(t2) :
    return False
  else :
    (t1L,k1,t1R) = exposeABT(t1)
    (t2L,k2,t2R) = exposeABT(t2)
    if (k1 != k2) :
      return False
    else :
      return ((isIsomorphic(t1L, t2L) and isIsomorphic(t1R,t2R))
              or
              (isIsomorphic(t1L, t2R) and isIsomorphic(t1R,t2L))
             )
```

ToC

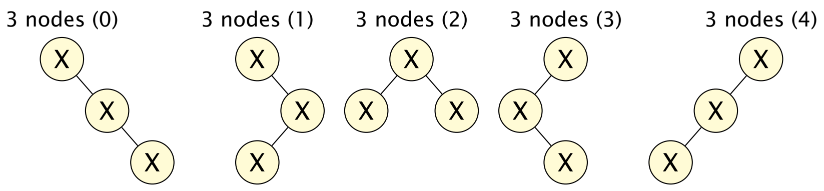
8.2 Generating Binary Trees

- The aim is to generate the shapes of all possible trees given a number of nodes

- First sketch a few trees to spot any pattern
- Write down a recurrence relation for the number of binary tree shapes with n nodes based on the number of tree shapes for less than n nodes
- Write a function `genBTs(x,n)` given a value `x` and an integer `n` generates the Python representation of all shapes of binary trees with `n` nodes with `x` at each node
- We first sketch a few trees to spot the pattern
- 0 nodes have 1 tree, `EmptyBT`, 1 node has 1 tree
- 2 nodes have 2 trees



- 3 nodes have 5 trees



- Let C_n be the number of binary tree shapes with n nodes then from the above diagrams we have:
- $C_0 = 1$
- $C_1 = 1$
- $C_2 = 2$
- $C_3 = 5$
- Eureka insight for a tree with n nodes if the left subtree has i nodes then the right subtree must have $n - i - 1$ nodes and i can range over 0 to $n - 1$
- The left and right subtrees must have C_i and C_{n-i-1} different possible shapes
- and there are n possible values for i from 0 to $n - 1$
- Hence $C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$
- $C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1} = \sum_{i=1}^{n} C_{i-1} C_{n-i}$
- Alternatively $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$
- Check $C_1 = C_0 C_0 = 1 \times 1 = 1$
- $C_2 = C_0 C_1 + C_1 C_0 = 1 \times 1 + 1 \times 1 = 2$

- $C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0 = 1 \times 2 + 1 \times 1 + 2 \times 1 = 5$
- $C_4 = C_0 C_3 + C_1 C_2 + C_2 C_1 + C_3 C_0$

  $= 1 \times 5 + 1 \times 2 + 2 \times 1 + 5 \times 1 = 14$
- The C_n are known as the Catalan numbers
- The simple recursive definition of genBTs follows from the recurrence relation directly
- Uses Python List Comprehensions see below
- This repeats the calculation of subtrees
- This is similar to the definition in Math.Combinat.Trees.Binary which is based on Knuth (2011, section 7.2.1.6),Knuth (1997, section 2.3.4.4)

```
def genABTs(x: T,n: int) -> [ABTree] :
  if n == 0 :
    return [mkEmptyABT()]
  elif n == 1 :
    return [mkNodeABT(x,mkEmptyABT(),mkEmptyABT())]
  else :
    ts = ([mkNodeABT(x,leftT,rightT)
           for (nu,nv) in splitsInt(n)
           for leftT   in genABTs(x,nu)
           for rightT  in genABTs(x,nv)])
    return ts

def splitsInt(n: int) -> [(int,int)] :
  prns = [(i, n - i - 1) for i in range(n)]
  return prns
```

- List comprehensions (tutorial), List comprehensions (reference) — a neat way of expressing iterations over a list, came from Miranda (see Wikipedia: List comprehension)
- Example: Square the even numbers between 0 and 9

```
Python3>>> [x ** 2 for x in range(10) if x % 2 == 0]
[0, 4, 16, 36, 64]
Python3>>> [(x,y) for x in range(4)
...               for y in range(4)
...               if x % 2 == 0
...                  and y % 3 == 0]
[(0, 0), (0, 3), (2, 0), (2, 3)]
Python3>>>
```

- In general

```
  [expr for target1 in iterable1 if cond1
        for target2 in iterable2 if cond2 ...
        for targetN in iterableN if condN ]
```

8.3 Catalan Numbers

- As with many other problems, it may be easier to find a recursive relation or recurrence for a problem and harder to find an efficient calculation.
- For the Catalan numbers it is possible to find a closed (non-recursive) expression for the Catalan numbers
- Below is a derivation of a closed expression — this is not part of M269 and is included for interest — the derivation uses a bit more Maths than the rest of these notes but it is explained as we progress

- This derivation is from Spivey (2019, page 208) and Wilf (1994, page 44)

8.3.1 Cauchy Product

- $\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{n=0}^{\infty} c_n x^n$

  where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

- The product forms a two-dimensional array — however we can arrange a sequence that goes through the array — see below and Spivak (2008, p486, p493, p513)

$$\begin{tikzcd}
a_0 b_0 \arrow[r] & a_0 b_1 \arrow[dl] & a_0 b_2 \arrow[r] & a_0 b_3 \arrow[dl] \\
a_1 b_0 \arrow[d] & a_1 b_1 \arrow[ur] & a_1 b_2 \arrow[dl] & \cdots \\
a_2 b_0 \arrow[ur] & a_2 b_1 \arrow[dl] & a_2 b_2 & \cdots \\
a_3 b_0 \arrow[d] & \cdots & \cdots & \\
\cdots & \cdots & \cdots &
\end{tikzcd}$$

- $\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right)$

  $= (a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$

  $= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$

- See Wikipedia: Cauchy product

- If we have $c(x) = \sum_{i=0}^{\infty} c_i x^i$

- $(c(x))^2 = \left(\sum_{i=0}^{\infty} c_i x^i\right)\left(\sum_{j=0}^{\infty} c_j x^j\right)$

  $= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} c_k c_{n-k}\right) x^n$

- This result is used in finding a closed form for the Catalan numbers

- Based on Mike Spivey 2013

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8.3.2 Catalan Recurrence

- $C_0 = 1$

- $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$

- Define $c(x)$ to be the generating function of the infinite sequence of the Catalan numbers
- $c(x) = \sum_{n=0}^{\infty} C_n x^n$
- Hence we can multiply both sides of the recurrence by x^n and sum
- $\sum_{n=0}^{\infty} C_{n+1} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n$
- $\sum_{n=0}^{\infty} C_{n+1} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n$
- $\frac{1}{x} \sum_{n=0}^{\infty} C_{n+1} x^{n+1} = (c(x))^2$ by Cauchy product
- $\frac{1}{x} \left(\sum_{n=0}^{\infty} C_n x^n - C_0 \right) = (c(x))^2$
- $\frac{1}{x}(c(x) - 1) = (c(x))^2$
- $x(c(x))^2 - c(x) + 1 = 0$
- $c(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$
- We know $c(0) = C_0 = 1$
- $c(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$
- We know $c(0) = C_0 = 1$
- Applying L'Hôpital's rule

  $$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$
- $\lim_{x \to 0^+} \frac{1 - \sqrt{1-4x}}{2x} = \lim_{x \to 0^+} \frac{2(1-4x)^{-\frac{1}{2}}}{2} = 1$
- Hence $c(x) = \frac{1 - \sqrt{1-4x}}{2x}$
- We now use the generalised Binomial theorem to expand this expression
- The generalised Binomial theorem has

  If $|x| > |y|$ and r is any complex number then

  $$(x+y)^r = \sum_{k-0}^{\infty} \binom{r}{k} x^{r-k} y^k$$

  where $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$

- $c(x) = \frac{1}{2x}(1 - \sqrt{1-4x}) = \frac{1}{2x}\left(1 - \sum_{n=0}^{\infty}\binom{1/2}{n}(-4x)^n\right)$
- The coefficient of x^n expands to

$$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}(-4)^n$$
$$= \frac{1(1-2)\cdots(1-2n+2)}{n!}(-1)^n 2^n$$

- The coefficient of x^n expands to

$$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}(-4)^n$$
$$= \frac{1(1-2)\cdots(1-2n+2)}{n!}(-1)^n 2^n$$

$$= \frac{(1)(3)\cdots(2n-3)(-1)^{n-1}}{(n!)^2}(-1)^n 2^n (n!)$$
$$= \frac{(1)(3)\cdots(2n-3)(-1)^{n-1}}{(n!)^2}(-1)^n (2n)(2n-2)\cdots(2)$$
$$= -\frac{(2n)!}{(n!)^2(2n-1)}$$
$$= -\binom{2n}{n}\frac{1}{2n-1}$$

- Hence $c(x) = \frac{1}{2x}\left(1 + \sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2n-1}x^n\right)$

$$= \frac{1}{2x}\left(1 + (-1) + \sum_{n=1}^{\infty}\binom{2n}{n}\frac{1}{2n-1}x^n\right)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{1}{2n-1}x^{n-1}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\binom{2(n+1)}{n+1}\frac{1}{2n+1}x^n$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(2n+2)(2n+1)}{(n+1)^2}\binom{2n}{n}\frac{1}{2n+1}x^n$$
$$= \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$$

- Hence $C_n = \frac{1}{n+1}\binom{2n}{n}$

8.3.3 Sample Catalan Numbers

```
  In[1]:= Series[(1 - Sqrt[1-4x])/(2x),{x,0,12}]
  Out[1]= SeriesData[x, 0, {1, 1, 2, 5, 14, 42, 132, 429, 1430, \
4862, 16796, 58786, 208012}, 0, 13, 1]
```

- Generating function form
- $1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6$
  $+ 429x^7 + 1430x^8 + 4862x^9 + 16796x^{10}$
  $+ 58786x^{11} + 208012x^{12} + O\left(x^{13}\right)$

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Commentary 6

6 Tutorial End, References and Colophon

- Future work and dates
- References to other Python texts or documentation
- References to other computing material
- Article version has the full references and bibliography with back references
- **Colophon**
- LaTeX with Beamer, Listings and other packages
- Index of Python code and diagrams
- PGF/TikZ for the diagrams
- External copies of the diagrams as PDF with tight bounding boxes are available

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9 Future Work

- Hashing and hash tables
- Binary search trees, height balanced binary search trees, AVL trees
- Graph algorithms
- Greedy algorithms
- Logic, Computability
- Future dates for tutorials and TMAs

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10 Web Sites & References

10.1 Python Web Links & References

- **Lutz (2013)** — one of the best introductory books

- **Lutz (2011)** — a more advanced book
- Lutz (2025) *Learning Python* — still one of the best introductory books
- Martelli et al. (2022)
- Ramalho (2022) a more advanced book
- **Python 3 Documentation** https://docs.python.org/3/
- **Python Style Guide PEP 8** https://www.python.org/dev/peps/pep-0008/ (Python Enhancement Proposals)

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10.2 Haskell Web Links & References

- **Haskell Language** https://www.haskell.org
- **HaskellWiki** https://wiki.haskell.org/Haskell
- **Learn You a Haskell for Great Good!** http://learnyouahaskell.com — very readable introduction to Haskell
- **Real World Haskell** http://book.realworldhaskell.org — more advanced
- **Thompson (2011)** — a good text for functional programming for beginners
- **Bird and Wadler (1988); Bird (1998, 2014)** — one of the best introductions but tough in parts, requires some mathematical maturity — the three books are in effect different editions
- Bird and Gibbons (2020) — developing the algorithms in a purely functional way
- **Functors, Applicatives, and Monads in Pictures** http://adit.io/posts/2013-04-17-functors,_applicatives,_and_monads_in_pictures.html — a very good outline with cartoons
- **Typeclassopedia** https://wiki.haskell.org/Typeclassopedia — a more formal introduction to Functors, Applicatives and Monads
- **Haskell Wikibook** https://en.wikibooks.org/wiki/Haskell

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10.3 Combinatorics

- Note that these references are not part of M269 and are here for reference
- Wikipedia: Twelvefold Way — advanced but a classification from Gian-Carlo Rota
- Combination
- Permutation

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Index

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Diagrams Index

Index for some of the PGF/TikZ diagrams. Note that the indexing commands are placed after the diagram code