M269 TMA03 Topics Revue

M269 Revue Tutorial

Contents

1 M269 End of Module Tutorial: Agenda

- Welcome & Introductions
- Topics from TMA03
- Abstract Data Types — Bags
- Abstract Data Types — Graphs
- Complexity
- Computability

Introductions — Me

- *Name* Phil Molyneux
- *Background* Physics and Maths, Operational Research, Computer Science
  - Undergraduate: Physics and Maths (Sussex)
  - Postgraduate: Physics (Sussex), Operational Research (Brunel), Computer Science (University College, London)
- *First programming languages* Fortran, BASIC, Pascal
- *Favourite Software*
  - Haskell — pure functional programming language
  - Text editors TextMate, Sublime Text — previously Emacs
  - Word processing and presentation slides in LaTeX
  - Mac OS X
- *Learning style* — I read the manual before using the software (really)

Introductions — You

- *Name* ?
- *Position in M269* ? Which part of which Units and/or Reader have you read ?
- *Particular topics you want to look at* ?
- *Learning Syle* ?

ToC

Commentary 1

1 Agenda, Aims and Topics

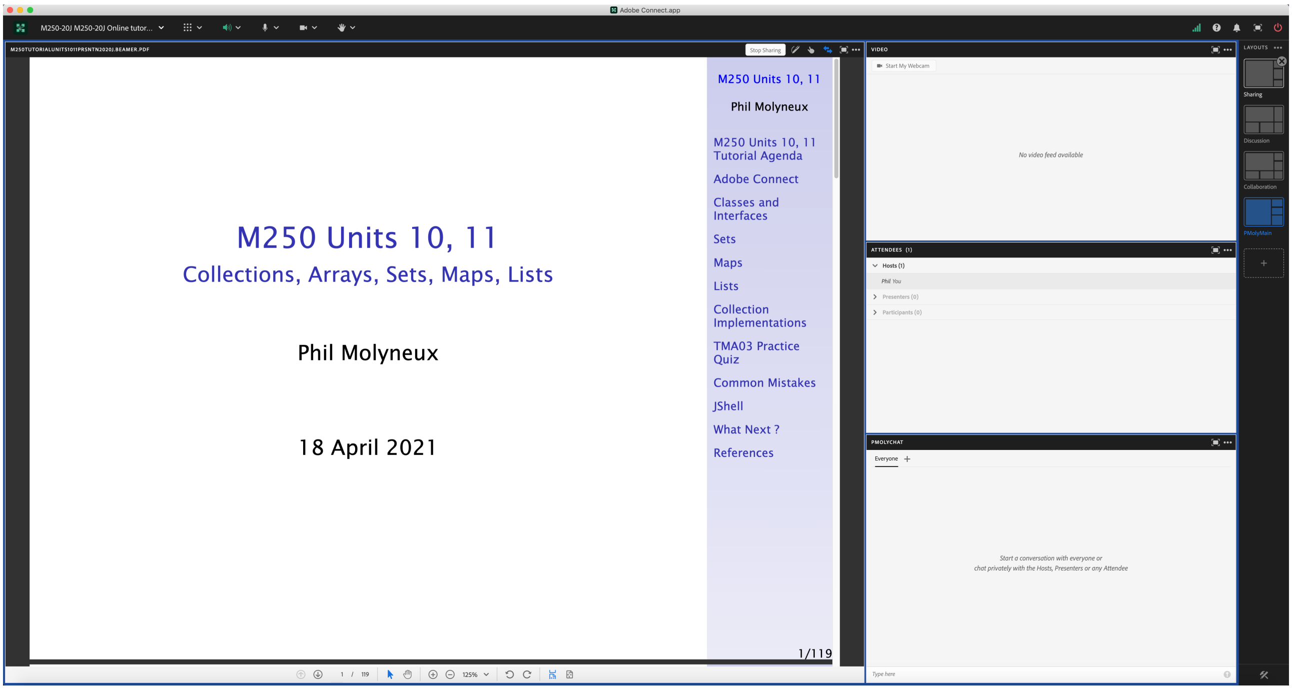
- Overview of aims of tutorial
- Note selection of topics
- Points about my own background and preferences
- Adobe Connect slides for reference
- Note that the Computability notes are here mainly for reference since the Complexity notes refer to them
- This session is mainly on the Complexity topics

ToC

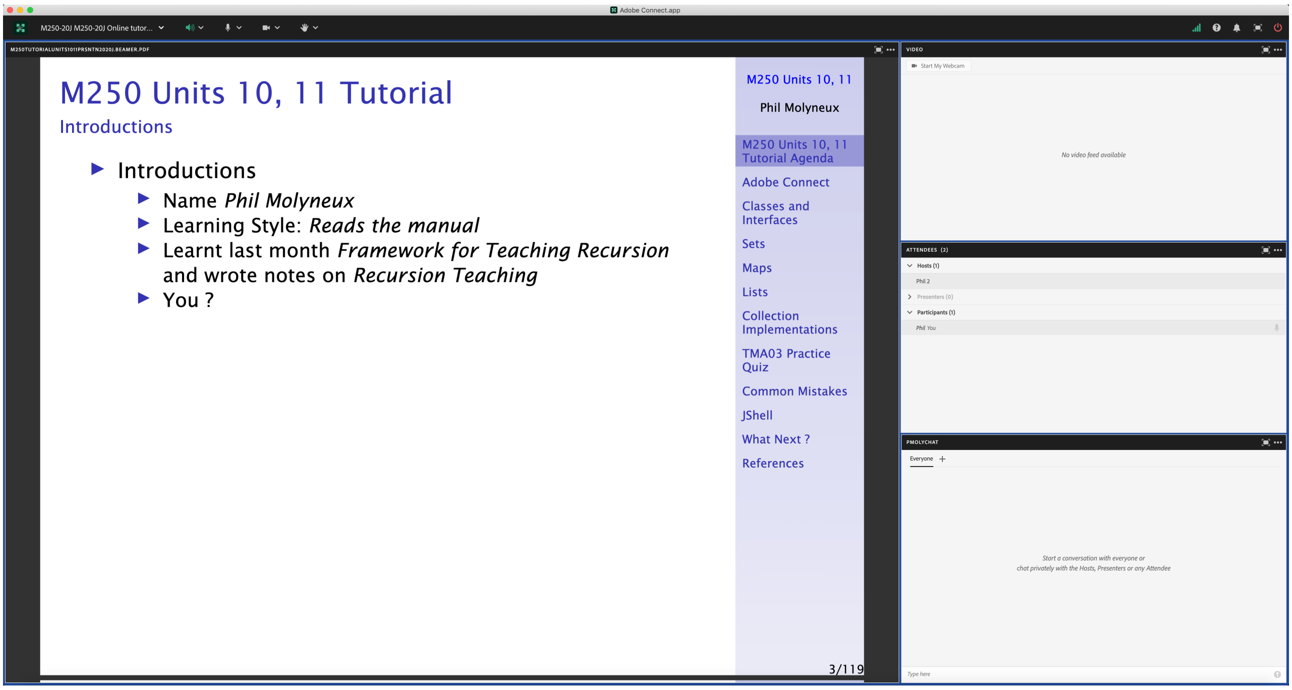
2 Adobe Connect Interface and Settings

2.1 Adobe Connect Interface

Adobe Connect Interface — Host View



Adobe Connect Interface — Participant View



2.2 Adobe Connect Settings

Adobe Connect — Settings

- **Everybody** *Menu bar* ❯ *Meeting* ❯ Speaker & Microphone Setup
- *Menu bar* ❯ *Microphone* ❯ Allow Participants to Use Microphone ✔
- Check Participants see the entire slide including slide numbers bottom right **Workaround**
  - *Disable Draw* *Share pod* ❯ *Menu bar* ❯ *Draw icon*
  - *Fit Width* *Share pod* ❯ *Bottom bar* ❯ *Fit Width icon* ✔

- *Meeting* › Preferences › General › **Host Cursor** › Show to all attendees
- *Menu bar* › *Video* › Enable **Webcam** for Participants ✔
- Do not *Enable single speaker mode*
- Cancel hand tool
- Do not enable green pointer
- **Recording** Meeting › Record Session ✔
- **Documents** Upload PDF with drag and drop to share pod
- Delete *Meeting* › Manage Meeting Information › Uploaded Content and *check filename* › *click on delete*

Adobe Connect — Access

- **Tutor Access**

  TutorHome › M269 Website › Tutorials

  Cluster Tutorials › M269 Online tutorial room

  Tutor Groups › M269 Online tutor group room

  Module-wide Tutorials › M269 Online module-wide room

- **Attendance**

  TutorHome › Students › View your tutorial timetables

- **Beamer Slide Scaling** 440% (422 x 563 mm)
- **Clear Everyone's Status**

  Attendee Pod › Menu › Clear Everyone's Status

- **Grant Access** and send link via email

  Meeting › Manage Access & Entry › Invite Participants...

- **Presenter Only Area**

  Meeting › Enable/Disable Presenter Only Area

Adobe Connect — Keystroke Shortcuts

- Keyboard shortcuts in Adobe Connect
- **Toggle Mic** ⌘ + M (Mac), Ctrl + M (Win) (On/Disconnect)
- **Toggle Raise-Hand status** ⌘ + E
- **Close dialog box** ⎋ (Mac), Esc (Win)
- **End meeting** ⌘ + \

2.3 Adobe Connect — Sharing Screen & Applications

- Share My Screen › Application tab › Terminal for Terminal
- Share menu › Change View › Zoom in for mismatch of screen size/resolution (Participants)

- (Presenter) Change to 75% and back to 100% (solves participants with smaller screen image overlap)
- Leave the application on the original display
- Beware blued hatched rectangles — from other (hidden) windows or contextual menus
- Presenter screen pointer affects viewer display — beware of moving the pointer away from the application
- First time: System Preferences › Security & Privacy › Privacy › Accessibility

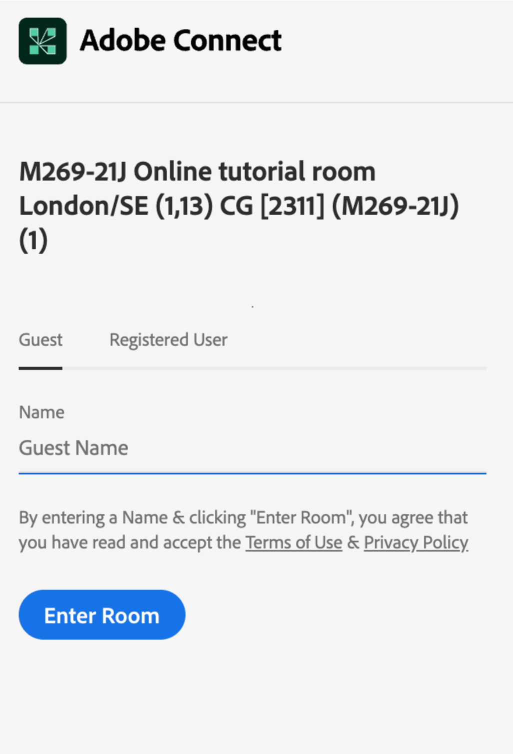
2.4 Adobe Connect — Ending a Meeting

- *Notes for the tutor only*
- **Student:** Meeting › Exit Adobe Connect
- **Tutor:**
- **Recording** Meeting › Stop Recording ✔
- **Remove Participants** Meeting › End Meeting... ✔
  - Dialog box allows for message with default message:
  - *The host has ended this meeting. Thank you for attending.*
- **Recording availability** *In course Web site for joining the room, click on the eye icon in the list of recordings under your recording* — edit description and name
- **Meeting Information** Meeting › Manage Meeting Information — can access a range of information in Web page.
- **Delete File Upload** Meeting › Manage Meeting Information › Uploaded Content tab select file(s) and click Delete
- **Attendance Report** see course Web site for joining room

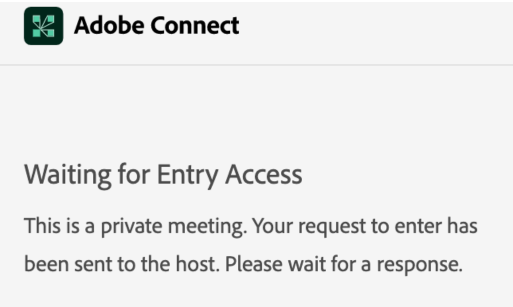
2.5 Adobe Connect — Invite Attendees

- **Provide Meeting URL** Menu › Meeting › Manage Access & Entry › Invite Participants...
- **Allow Access without Dialog** Menu › Meeting › Manage Meeting Information provides new browser window with *Meeting Information* Tab bar › Edit Information
- Check *Anyone who has the URL for the meeting can enter the room*
- Default *Only registered users and accepted guests may enter the room*
- **Reverts to default next session but URL is fixed**
- Guests have blue icon top, registered participants have yellow icon top — same icon if URL is open
- See Start, attend, and manage Adobe Connect meetings and sessions
- Click on the link sent in email from the Host
- Get the following on a Web page

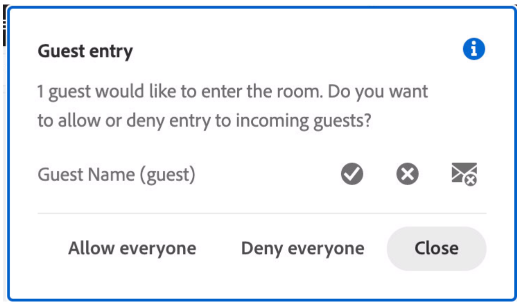
- As *Guest* enter your name and click on Enter Room



- See the *Waiting for Entry Access* for *Host* to give permission



- *Host* sees the following dialog in *Adobe Connect* and grants access



2.6 Layouts

- **Creating new layouts** example *Sharing* layout
- Menu 〉Layouts 〉Create New Layout... Create a New Layout dialog 〉Create a new blank layout and name it *PMolyMain*
- New layout has no Pods but does have Layouts Bar open (see Layouts menu)

- **Pods**
- Menu › Pods › Share › Add New Share and resize/position — initial name is *Share n* — rename *PMolyShare*
- **Rename Pod** Menu › Pods › Manage Pods. . . Manage Pods › Select › Rename or Double-click & rename
- Add Video pod and resize/reposition
- Add Attendance pod and resize/reposition
- Add Chat pod — rename it *PMolyChat* — and resize/reposition
- Dimensions of **Sharing** layout (on 27-inch iMac)
  - Width of Video, Attendees, Chat column 14 cm
  - Height of Video pod 9 cm
  - Height of Attendees pod 12 cm
  - Height of Chat pod 8 cm
- **Duplicating Layouts** does *not* give new instances of the Pods and is probably not a good idea (apart from local use to avoid delay in reloading Pods)
- **Auxiliary Layouts** name *PMolyAux0n*
  - Create new Share pod
  - Use existing Chat pod
  - Use same Video and Attendance pods

2.7 Chat Pods

- **Format Chat text**
- Chat Pod › menu icon › My Chat Color
- Choices: Red, Orange, Green, Brown, Purple, Pink, Blue, Black
- Note: Color reverts to Black if you switch layouts
- Chat Pod › menu icon › Show Timestamps

2.8 Graphics Conversion for Web

- Conversion of the screen snaps for the installation of Anaconda on 1 May 2020
- Using GraphicConverter 11
- File › Convert & Modify › Conversion › Convert
- Select files to convert and destination folder
- Click on Start selected Function or ⌘ + ⏎

2.9 Adobe Connect Recordings

- *Menu bar* › *Meeting* › Preferences › Video
- Aspect ratio › Standard (4:3) (not Wide screen (16:9) default)
- Video quality › Full HD (1080p not High default 480p)
- **Recording** *Menu bar* › *Meeting* › Record Session ✔
- **Export Recording**
- *Menu bar* › *Meeting* › Manage Meeting Information
- *New window* › Recordings › *check Tutorial* › Access Type button
- check Public › check Allow viewers to download
- **Download Recording**
- *New window* › Recordings › *check Tutorial* › Actions › Download File

Commentary 2

2 Computability

- Description of Turing Machine
- Turing Machine examples
- Computability, Decidability and Algorithms
- Non-computability — Halting Problem
- Reductions and non-computability
- Lambda Calculus (optional)
- Note that the Computability notes are here mainly for reference since the Complexity notes refer to them
- This session is mainly on the Complexity topics

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3 Computability

Ideas of Computation

- The idea of an algorithm and what is effectively computable
- **Church-Turing thesis** Every function that would naturally be regarded as computable can be computed by a deterministic Turing Machine. (Unit 7 Section 4)
- See Phil Wadler on computability theory performed as part of the Bright Club at The Strand in Edinburgh, Tuesday 28 April 2015

Computability — Models of Computation

- In automata theory, a *problem* is the question of deciding whether a given string is a member of some particular language

- If Σ is an alphabet, and L is a language over Σ, that is $L \subseteq \Sigma^*$, where Σ^* is the set of all strings over the alphabet Σ then we have a more formal definition of *decision problem*
- Given a string $w \in \Sigma^*$, decide whether $w \in L$
- Example: Testing for a prime number — can be expressed as the language L_p consisting of all binary strings whose value as a binary number is a prime number (only divisible by 1 or itself)
- See Hopcroft et al. (2007, section 1.5.4)

Automata Theory — Alphabets, Strings, Languages

- An **Alphabet**, Σ, is a finite, non-empty set of symbols.
- Binary alphabet $\Sigma = \{0, 1\}$
- Lower case letters $\Sigma = \{a, b, \ldots, z\}$
- A **String** is a finite sequence of symbols from some alphabet
- 01101 is a string from the Binary alphabet $\Sigma = \{0, 1\}$
- The **Empty string**, ϵ, contains no symbols
- **Powers**: Σ^k is the set of strings of length k with symbols from Σ
- The set of all strings over an alphabet Σ is denoted Σ^*
- $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$
- **Question** Does $\Sigma^0 = \varnothing$? ($\varnothing$ is the empty set)
- An **Language**, L, is a subset of Σ^*
- The set of binary numerals whose value is a prime

  $\{10, 11, 101, 111, 1011, \ldots\}$
- The set of binary numerals whose value is a square

  $\{100, 1001, 10000, 11001, \ldots\}$

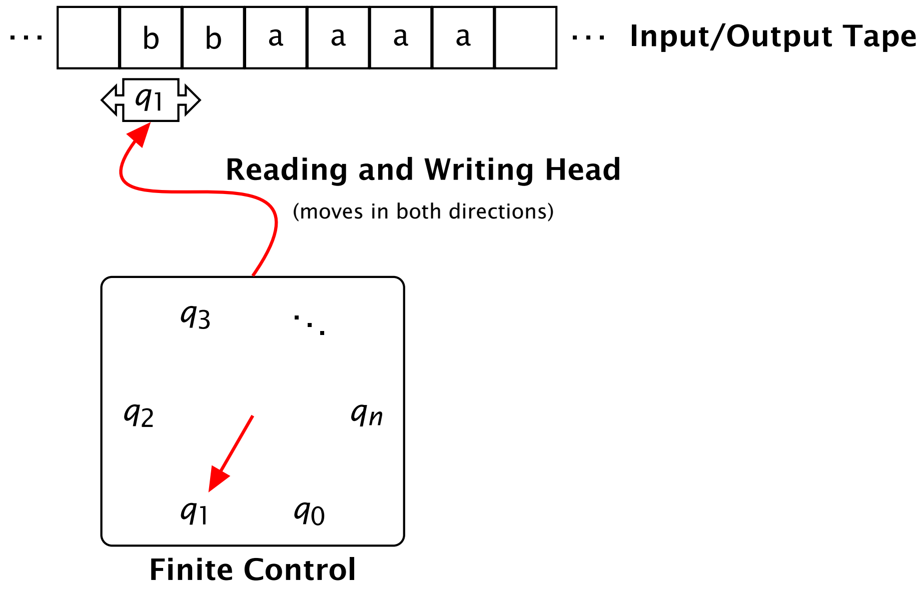
Computability — Church-Turing Thesis

- **Church-Turing thesis** Every function that would naturally be regarded as computable can be computed by a deterministic Turing Machine.
- **physical Church-Turing thesis** Any finite physical system can be simulated (to any degree of approximation) by a Universal Turing Machine.
- **strong Church-Turing thesis** Any finite physical system can be simulated (to any degree of approximation) with polynomial slowdown by a Universal Turing Machine.
- Shor's algorithm (1994) — quantum algorithm for factoring integers — an NP problem that is not known to be P — also not known to be NP-complete and we have no proof that it is not in P
- Reference: Section 4 of Unit 6 & 7 Reader

3.1 The Turing Machine

- **Finite control** which can be in any of a finite number of *states*
- **Tape** divided into cells, each of which can hold one of a finite number of symbols
- Initially, the **input**, which is a finite-length string of symbols in the *input alphabet*, is placed on the tape
- All other tape cells (extending unbounded left and right) hold a special symbol called *blank*
- A **tape head** which initially is over the leftmost input symbol
- A **move** of the Turing Machine depends on the state and the tape symbol scanned
- A move can change state, write a symbol in the current cell, move left, right or stay
- References: Hopcroft et al. (2007, page 326), Unit 6 & 7 Reader (section 5.3)

Turing Machine Diagram



Source: Sebastian Sardina http://www.texample.net/tikz/examples/turing-machine-2/
Date: 18 February 2012 (seen Sunday, 24 August 2014)
Further Source: Partly based on Ludger Humbert's pics of Universal Turing Machine at https://haspe.homeip.net/projekte/ddi/browser/tex/pgf2/turingmaschine-schema.tex (not found) — http://www.texample.net/tikz/examples/turing-machine/

Turing Machine notation

- Q finite set of states of the finite control
- Σ finite set of *input symbols* (M269 S)
- Γ complete set of *tape symbols* $\Sigma \subset \Gamma$
- δ Transition function (M269 instructions, I)
  $\delta :: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$
  $\delta(q, X) \mapsto (p, Y, D)$

- $\delta(q, X)$ takes a state, q and a tape symbol, X and returns (p, Y, D) where p is a state, Y is a tape symbol to overwrite the current cell, D is a direction, Left, Right or Stay
- q_0 *start state* $q_0 \in Q$
- B *blank symbol* $B \in \Gamma$ and $B \notin \Sigma$
- F set of *final or accepting states* $F \subseteq Q$

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3.2 Turing Machine Examples

Turing Machine Simulators

- Morphett's Turing machine simulator — the examples below are adapted from here
- Ugarte's Turing machine simulator
- XKCD A Bunch of Rocks — XKCD Explanation

  Image below (will need expanding to be readable)
- The term *state* is used in two different ways:

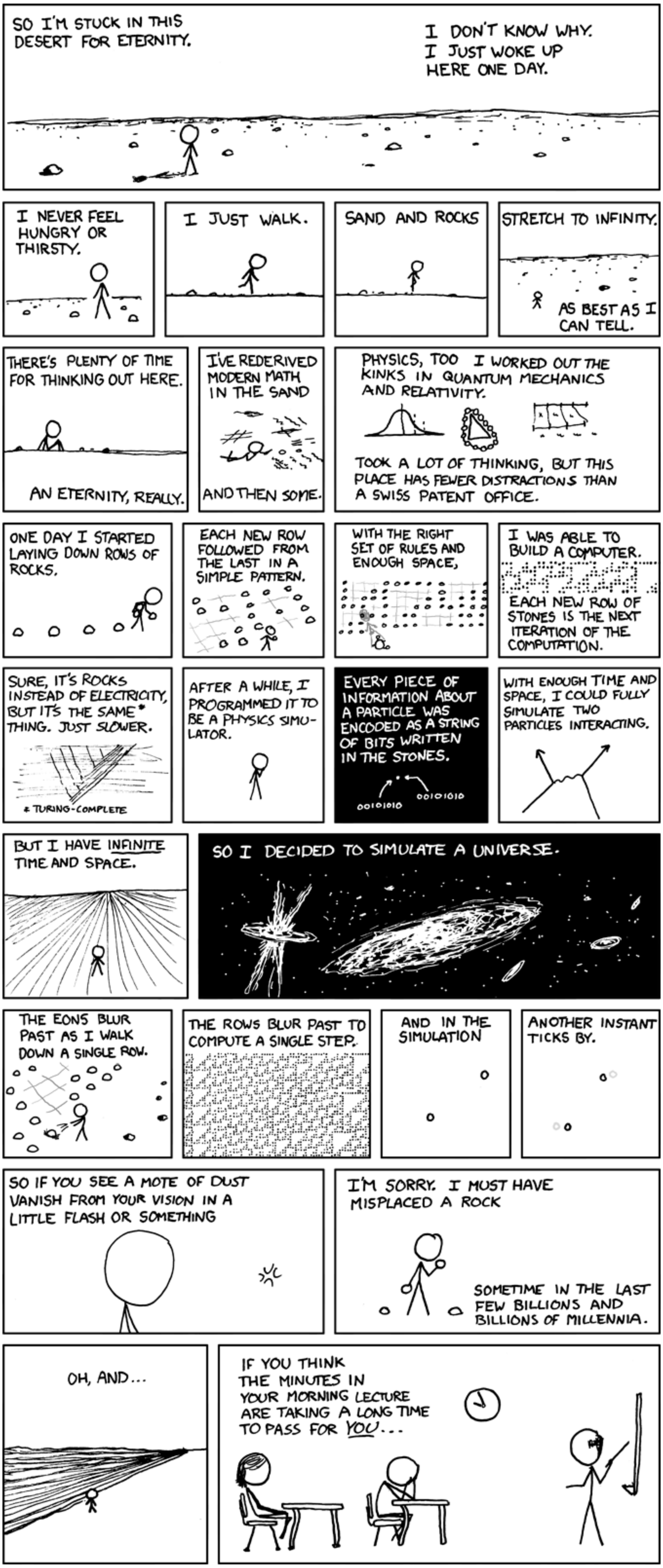
  The value of the *Finite Control*

  The overall configuration of *Finite Control* and current contents of the tape

  See Turing Machine: State

  will lead to some confusion

XKCD A Bunch of Rocks


SO I'M STUCK IN THIS DESERT FOR ETERNITY.
I DON'T KNOW WHY. I JUST WOKE UP HERE ONE DAY.
I NEVER FEEL HUNGRY OR THIRSTY.
I JUST WALK.
SAND AND ROCKS
STRETCH TO INFINITY.
AS BEST AS I CAN TELL.
THERE'S PLENTY OF TIME FOR THINKING OUT HERE.
AN ETERNITY, REALLY.
I'VE REDERIVED MODERN MATH IN THE SAND
AND THEN SOME.
PHYSICS, TOO. I WORKED OUT THE KINKS IN QUANTUM MECHANICS AND RELATIVITY.
TOOK A LOT OF THINKING, BUT THIS PLACE HAS FEWER DISTRACTIONS THAN A SWISS PATENT OFFICE.
ONE DAY I STARTED LAYING DOWN ROWS OF ROCKS.
EACH NEW ROW FOLLOWED FROM THE LAST IN A SIMPLE PATTERN.
WITH THE RIGHT SET OF RULES AND ENOUGH SPACE,
I WAS ABLE TO BUILD A COMPUTER.
EACH NEW ROW OF STONES IS THE NEXT ITERATION OF THE COMPUTATION.
SURE, IT'S ROCKS INSTEAD OF ELECTRICITY, BUT IT'S THE SAME* THING. JUST SLOWER.
* TURING-COMPLETE
AFTER A WHILE, I PROGRAMMED IT TO BE A PHYSICS SIMULATOR.
EVERY PIECE OF INFORMATION ABOUT A PARTICLE WAS ENCODED AS A STRING OF BITS WRITTEN IN THE STONES.
00101010
00101010
WITH ENOUGH TIME AND SPACE, I COULD FULLY SIMULATE TWO PARTICLES INTERACTING.
BUT I HAVE INFINITE TIME AND SPACE.
SO I DECIDED TO SIMULATE A UNIVERSE.
THE EONS BLUR PAST AS I WALK DOWN A SINGLE ROW.
THE ROWS BLUR PAST TO COMPUTE A SINGLE STEP.
AND IN THE SIMULATION
ANOTHER INSTANT TICKS BY.
SO IF YOU SEE A MOTE OF DUST VANISH FROM YOUR VISION IN A LITTLE FLASH OR SOMETHING
I'M SORRY. I MUST HAVE MISPLACED A ROCK
SOMETIME IN THE LAST FEW BILLIONS AND BILLIONS OF MILLENNIA.
OH, AND...
IF YOU THINK THE MINUTES IN YOUR MORNING LECTURE ARE TAKING A LONG TIME TO PASS FOR YOU...

Turing Machine Examples: Meta-Exercise

- For each of the Turing Machine Examples below, identify

  $(Q, \Sigma, \Gamma, \delta, q_0, B, F)$

3.2.1 The Successor Function

- **Input** binary representation of numeral n
- **Output** binary representation of $n+1$
- Example $1010 \mapsto 1011$ and $1011 \mapsto 1100$
- Initial cell: leftmost symbol of n
- **Strategy**
- **Stage A** make the rightmost cell the current cell
- **Stage B** Add 1 to the current cell.
- If the current cell is 0 then replace it with 1 and go to stage C
- If the current cell is 1 replace it with 0 and go to stage B and move Left
- If the current cell is blank, replace it by 1 and go to stage C
- **Stage C** Finish up by making the leftmost cell current
- Represent the Turing Machine program as a list of quintuples (q, X, p, Y, D)
- **Stage A**

  $(q_0, 0, q_0, 0, R)$

  $(q_0, 1, q_0, 1, R)$

  (q_0, B, q_1, B, L)

- **Stage B**

  $(q_1, 0, q_2, 1, S)$

  $(q_1, 1, q_1, 0, L)$

  $(q_1, B, q_2, 1, S)$

- **Stage C**

  $(q_2, 0, q_2, 0, L)$

  $(q_2, 1, q_2, 1, L)$

  (q_2, B, q_h, B, R)

(Smith, 2013, page 315)

- **Exercise** Translate the quintuples (q, X, p, Y, D) into English and check they are the same as the specification
- **Stage A** make the rightmost cell the current cell

  $(q_0, 0, q_0, 0, R)$

  If state q_0 and read symbol 0 then stay in state q_0 write 0, move R

$(q_0, 1, q_0, 1, R)$

If state q_0 and read symbol 1 then stay in state q_0 write 1, move R

(q_0, B, q_1, B, L)

If state q_0 and read symbol B then state q_1 write B, move L

- **Exercise** Translate the quintuples (q, X, p, Y, D) into English
- **Stage B** Add 1 to the current cell.

  $(q_1, 0, q_2, 1, S)$

  If state q_1 and read symbol 0 then state q_2 write 1, stay

  $(q_1, 1, q_1, 0, L)$

  If state q_1 and read symbol 1 then state q_1 write 0, move L

  $(q_1, B, q_2, 1, S)$

  If state q_1 and read symbol B then state q_2 write 1, stay
- **Exercise** Translate the quintuples (q, X, p, Y, D) into English
- **Stage C** Finish up by making the leftmost cell current

  $(q_2, 0, q_2, 0, L)$

  If state q_2 and read symbol 0 then state q_2 write 0, move L

  $(q_2, 1, q_2, 1, L)$

  If state q_2 and read symbol 1 then state q_2 write 0, move L

  (q_2, B, q_h, B, R)

  If state q_2 and read symbol B then state q_h write B, move R HALT
- Notice that the Turing Machine feels like a series of `if ... then` or `case` statements inside a `while` loop

Turing Machine Examples: Meta-Exercise: Successor Function

- Identify $(Q, \Sigma, \Gamma, \delta, q_0, B, F)$ *Blank slide for working*
- Identify $(Q, \Sigma, \Gamma, \delta, q_0, B, F)$
- $Q = \{q_0, q_1, q_2, q_h\}$
- q_0 finding the rightmost symbol
- q_1 add 1 to current cell
- q_2 move to leftmost cell
- q_h finish
- $\Sigma = \{0, 1\}$
- $\Gamma = \Sigma \cup \{B\}$
- $\delta :: Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$

  $\delta(q, X) \mapsto (p, Y, D)$

δ is represented as {(q,X,p,Y,D)}

equivalent to $\{((q,X),(p,Y,D))\}$ *set of pairs*

- q_0 start with leftmost symbol under head, state moving to rightmost symbol
- B is ␣ a visible space
- $F = \{q_h\}$
- **Sample Evaluation** $11 \mapsto 100$
- **Representation** $\cdots BX_1X_2\cdots X_{i-1}qX_iX_{i+1}\cdots X_nB\cdots$

  $q_0 11$

  $1q_0 1$

  $11q_0 B$

  $1q_1 1$

  $q_1 10$

  $q_1 B00$

  $q_2 100$

  $q_2 B100$

  $q_h 100$
- **Exercise** evaluate $1011 \mapsto 1100$
- **Representation** $\cdots BX_1X_2\cdots X_{i-1}qX_iX_{i+1}\cdots X_nB\cdots$
- q is the state of the TM
- The head is scanning the symbol X_i
- Leading or trailing blanks B are (usually) not shown unless the head is scanning them
- $\vdash_M$ denotes one move of the TM M
- $\vdash_M^*$ denotes zero or more moves
- $\vdash$ will be used if the TM M is understood
- If (q, X_i, p, Y, L) denotes a TM move then

  $X_1\cdots X_{i-1}qX_i\cdots X_n \vdash_M X_1\cdots X_{i-2}pX_{i-1}Y\cdots X_n$

(Hopcroft et al., 2007, sec 8.2.3)

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3.2.2 The Binary Palindrome Function

- **Input** binary string s
- **Output** YES if palindrome, NO otherwise
- Example $1010 \mapsto NO$ and $1001 \mapsto YES$
- Initial cell: leftmost symbol of s

- **Strategy**
- **Stage A** read the leftmost symbol
- If blank then accept it and go to stage D otherwise erase it
- **Stage B** find the rightmost symbol
- If the current cell matches leftmost recently read then erase it and go to stage C
- Otherwise reject it and go to stage E
- **Stage C** return to the leftmost symbol and stage A
- **Stage D** print YES and halt
- **Stage E** erase the remaining string and print NO
- Represent the Turing Machine program as a list of quintuples (q, X, p, Y, D)
- **Stage A** read the leftmost symbol

  $(q_0, 0, q_{1_o}, B, R)$

  $(q_0, 1, q_{1_i}, B, R)$

  (q_0, B, q_5, B, S)
- **Stage B** find rightmost symbol

  $(q_{1_o}, B, q_{2_o}, B, L)$

  $(q_{1_o}, *, q_{1_o}, *, R)$ * is a wild card, matches anything

  $(q_{1_i}, B, q_{2_i}, B, L)$

  $(q_{1_i}, *, q_{1_i}, *, R)$
- **Stage B** check

  $(q_{2_o}, 0, q_3, B, L)$

  (q_{2_o}, B, q_5, B, S)

  $(q_{2_o}, *, q_6, *, S)$

  $(q_{2_i}, 1, q_3, B, L)$

  (q_{2_i}, B, q_5, B, S)

  $(q_{2_i}, *, q_6, *, S)$
- **Stage C** return to the leftmost symbol and stage A

  (q_3, B, q_5, B, S)

  $(q_3, *, q_4, *, L)$

  (q_4, B, q_0, B, R)

  $(q_4, *, q_4, *, L)$
- **Stage D** accept and print YES

  $(q_5, *, q_{5_a}, Y, R)$

  $(q_{5_a}, *, q_{5_b}, E, R)$

$(q5_b, *, q_7, S, S)$

- **Stage E** erase the remaining string and print NO

  $(q_6, B, q6_a, N, R)$

  $(q_6, *, q_6, B, L)$

  $(q6_a, *, q_7, O, S)$

- Finish

  (q_7, B, q_h, B, R)

  $(q_7, *, q_7, *, L)$

Turing Machine Examples: Meta-Exercise: Binary Palindrome Function

- Identify $(Q, \Sigma, \Gamma, \delta, q_0, B, F)$ *Blank slide for working*
- Identify $(Q, \Sigma, \Gamma, \delta, q_0, B, F)$
- $Q = \{q_0, q1_o, q1_i, q2_o, q2_i, q_3, q_4, q_5, q5_a, q5_b, q_6, q6_a, q_7, q_h\}$
- q_0 read leftmost symbol
- $q1_o, q1_i$ find rightmost symbol looking for 0 or 1
- $q2_o, q2_i$ check, confirm or reject
- q_3, q_4 check finish or move to start
- q_5, q_6, q_7 print YES or NO and finish
- q_h finish
- $\Sigma = \{0, 1\}$
- $\Gamma = \Sigma \cup \{B, Y, E, S, N, O\}$
- $\delta :: Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$

  $\delta(q, X) \mapsto (p, Y, D)$

  δ is represented as {(q,X,p,Y,D)}

  equivalent to $\{((q, X), (p, Y, D))\}$ *set of pairs*

- Start with leftmost symbol under head, state q_0
- B is ␣ a visible space
- $F = \{q_h\}$
- **Sample Evaluation** 101 $\mapsto$ YES

  $q_0 101 \vdash Bq1_i 01 \vdash B0q1_i 1 \vdash B01q1_i B$

  $\vdash B0q2_i 1$

  $\vdash Bq_3 0B \vdash q_4 B0B$

  $\vdash Bq_0 0B \vdash BBq1_o B$

  $\vdash Bq2_o BB$

$\vdash Bq_5BB \vdash Yq_{5_a}B \vdash YEq_{5_b}B \vdash YEq_7S$

$\vdash Yq_7ES \vdash Bq_7YES \vdash q_7BYES \vdash q_hYES$

- **Exercise** Evaluate 110 $\mapsto$ NO

ToC

3.2.3 Binary Addition Example

- **Input** two binary numerals separated by a single space $n1\ \ n2$
- **Output** binary numeral which is the sum of the inputs
- Example $110110 + 101011 \mapsto 1100001$
- Initial cell: leftmost symbol of $n1\ \ n2$
- **Insight** look at the arithmetic algorithm

␣	1	1	0	1	1	0
␣	1	0	1	0	1	1
1	1	0	0	0	0	1

- **Discussion** how can we overwrite the first number with the result and remember how far we have gone ?

Binary Addition Example — Arithmetic Reinvented

␣	1	1	0	1	1	0
␣	1	0	1	0	1	1
␣	1	1	0	1	1	y
␣	1	0	1	0	1	␣
␣	1	1	1	0	x	y
␣	1	0	1	0	␣	␣
␣	1	1	1	x	x	y
␣	1	0	1	␣	␣	␣
1	0	0	x	x	x	y
␣	1	0	␣	␣	␣	␣
1	0	x	x	x	x	y
␣	1	␣	␣	␣	␣	␣
1	y	x	x	x	x	y
␣	␣	␣	␣	␣	␣	␣
1	1	0	0	0	0	1

- **Input** two binary numerals separated by a single space $n1\ \ n2$
- **Output** binary numeral which is the sum of the inputs
- Example $110110 + 101011 \mapsto 1100001$
- Initial cell: leftmost symbol of $n1\ \ n2$
- **Strategy**

- **Stage A** find the rightmost symbol

  If the symbol is 0 erase and go to stage Bx

  If the symbol is 1 erase go to stage By

  If the symbol is blank go to stage F

  dealing with each digit in $n2$

  if no further digits in $n2$ go to final stage

- **Stage Bx** Move left to a blank go to stage Cx
- **Stage By** Move left to a blank go to stage Cy

  moving to $n1$

- **Stage Cx** Move left to find first 0, 1 or B

  Turn 0 or B to X, turn 1 to Y and go to stage A

  adding 0 to a digit finalises the result (no *carry one*)

- **Stage Cy** Move left to find first 0, 1 or B

  Turn 0 or B to 1 and go to stage D

  Turn 1 to 0, move left and go to stage Cy

  dealing with the *carry one* in school arithmetic

- **Stage D** move right to X, Y or B and go to stage E
- **Stage E** replace 0 by X, 1 by Y, move right and go to Stage A

  finalising the value of a digit resulting from a *carry*

- **Stage F** move left and replace X by 0, Y by 1 and at B halt
- Represent the Turing Machine program as a list of quintuples (q, X, p, Y, D)
- **Stage A** find the rightmost symbol

  (q_0, B, q_1, B, R)

  $(q_0, *, q_0, *, R)$ * is a wild card, matches anything

  (q_1, B, q_2, B, L)

  $(q_1, *, q_1, *, R)$

  $(q_2, 0, q_{3_x}, B, L)$

  $(q_2, 1, q_{3_y}, B, L)$

  (q_2, B, q_7, B, L)

- **Stage Bx** move left to blank

  $(q_{3_x}, B, q_{4_x}, B, L)$

  $(q_{3_x}, *, q_{3_x}, *, L)$

- **Stage By** move left to blank

  $(q_{3_y}, B, q_{4_y}, B, L)$

$(q_{3_y}, *, q_{3_y}, *, L)$

- **Stage Cx** move left to 0, 1, or blank

  $(q_{4_x}, 0, q_0, x, R)$

  $(q_{4_x}, 1, q_0, y, R)$

  (q_{4_x}, B, q_0, x, R)

  $(q_{4_x}, *, q_{4_x}, *, L)$

- **Stage Cy** move left to 0, 1, or blank

  $(q_{4_y}, 0, q_5, 1, S)$

  $(q_{4_y}, 1, q_{4_y}, 0, L)$

  $(q_{4_y}, B, q_5, 1, S)$

  $(q_{4_y}, *, q_{4_y}, *, L)$

- **Stage D** move right to x, y or B

  (q_5, x, q_6, x, L)

  (q_5, y, q_6, y, L)

  (q_5, B, q_6, B, L)

  $(q_5, *, q_5, *, R)$

- **Stage E** replace 0 by x, 1 by y

  $(q_6, 0, q_0, x, R)$

  $(q_6, 1, q_0, y, R)$

- **Stage F** replace x by 0, y by 1

  $(q_7, x, q_7, 0, L)$

  $(q_7, y, q_7, 1, L)$

  (q_7, B, q_h, B, R)

  $(q_7, *, q_7, *, L)$

- **Exercise** Evaluate $11 + 10 \mapsto 101$

Turing Machine Examples: Meta-Exercise: Successor Function

- Identify $(Q, \Sigma, \Gamma, \delta, q_0, B, F)$ *Blank slide for working*
- Identify $(Q, \Sigma, \Gamma, \delta, q_0, B, F)$
- $Q = \{q_0, q_1, q_2, q_{3_x}, q_{3_y}, q_{4_x}, q_{4_y}, q_5, q_6, q_7, q_h\}$
- q_0, q_1, q_2 find rightmost symbol of second number
- q_{3_x}, q_{3_y} move left to inter-number blank
- q_{4_x}, q_{4_y} move left to 0, 1 or blank
- q_5 move right to x, y or B

- q_6 replace 0 by x, 1 by y and move right
- q_7 replace x by 0, y by 1 and move left
- q_h finish
- $\Sigma = \{0, 1\}$
- $\Gamma = \Sigma \cup \{B, x, y\}$
- $\delta :: Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$

  $\delta(q, X) \mapsto (p, Y, D)$

  δ is represented as $\{(q, X, p, Y, D)\}$

  equivalent to $\{((q, X), (p, Y, D))\}$ *set of pairs*
- Start with leftmost symbol under head, state q_0
- B is ␣ a visible space
- $F = \{q_h\}$
- **Exercise** Evaluate $11 + 10 \mapsto 101$
- **Stage A** find the rightmost symbol

  $BBq_0 11B10B$ Note space symbols B at start and end

$\vdash BB1q_0 1B10B$

$\vdash BB11q_0 B10B$

$\vdash BB11Bq_1 10B$

$\vdash BB11B1q_1 0B$

$\vdash BB11B10q_1 B$

$\vdash BB11B1q_2 0B$

$\vdash BB11Bq_{3_x} 1BB$

- **Stage Bx** move left to blank

$\vdash B11q_{3_x} B1BB$

- **Stage Cx** move left to 0, 1, or blank

$\vdash BB1q_{4_x} 1B1BB$

$\vdash BB1Yq_0 B1BB$

- **Stage A** find the rightmost symbol

$\vdash BB1BYBq_1 1BB$

$\vdash BB1YB1q_1 BB$

$\vdash BB1YBq_2 1BB$

$\vdash BB1Yq_{3_y} BBBB$

- **Stage Cy** move left to 0, 1, or blank

$\vdash BB1q_{4_y} YBBBB$

$\vdash BBq4_y1YBBBB$

$\vdash Bq4_yB0YBBBB$

$\vdash Bq_5 10YBBBB$

- **Stage D** move right to x, y or B

$\vdash Bq_5 0YBBBB$

$\vdash B0q_5 YBBBB$

$\vdash Bq_6 0YBBBB$

- **Stage E** replace 0 by x, 1 by y

$\vdash B1Xq_0 YBBBB$

- **Stage A** find the rightmost symbol

$\vdash B1XYq_0 BBBB$

$\vdash B1XYBq_1 BBB$

$\vdash B1XYq_2 BBBB$

$\vdash B1Xq_7 YBBBB$

- **Stage F** replace x by 0, y by 1

$\vdash B1q_7X1BBBB$

$\vdash Bq_7 101BBBB$

$\vdash Bq_7B101BBBB$

$\vdash Bq_h 101BBBB$

- This is mimicking what you learnt to do on paper as a child! Real *step-by-step* instructions
- See Morphett's Turing machine simulator for more examples (takes too long by hand!)

ToC

3.3 Computability, Decidability and Algorithms

Universal Turing Machine

- Universal Turing Machine, U, is a Turing Machine that can simulate any arbitrary Turing machine, M
- Achieves this by encoding the transition function of M in some standard way
- The input to U is the encoding for M followed by the data for M
- See Turing machine examples
- **Decidable** — there is a TM that will halt with yes/no for a decision problem — that is, given a string w over the alphabet of P the TM with halt and return yes.no the string is in the language P (same as *recursive* in Recursion theory — old use of the word)

- **Semi-decidable** — there is a TM will halt with yes if some string is in *P* but may loop forever on some inputs (same as *recursively enumerable*) — *Halting Problem*
- **Highly-undecidable** — no outcome for any input — *Totality, Equivalence Problems*

Undecidable Problems

- **Halting problem** — the problem of deciding, given a program and an input, whether the program will eventually halt with that input, or will run forever — term first used by Martin Davis 1952
- **Entscheidungsproblem** — the problem of deciding whether a given statement is provable from the axioms using the rules of logic — shown to be undecidable by Turing (1936) by reduction from the *Halting problem* to it
- **Type inference and type checking** in the second-order lambda calculus (important for functional programmers, Haskell, GHC implementation)
- Undecidable problem — see link to list

(Turing, 1936, 1937)

3.3.1 Non-Computability — Halting Problem

Halting Problem — Sketch Proof

- Halting problem — is there a program that can determine if any arbitrary program will halt or continue forever ?
- Assume we have such a program (Turing Machine) `h(f,x)` that takes a program `f` and input `x` and determines if it halts or not

```
h(f,x)
  = if f(x) runs forever
      return True
    else
      return False
```

- We shall prove this cannot exist by contradiction
- Now invent two further programs:
- `q(f)` that takes a program `f` and runs `h` with the input to `f` being a copy of `f`
- `r(f)` that runs `q(f)` and halts if `q(f)` returns `True`, otherwise it loops

```
q(f)
  = h(f,f)

r(f)
  = if q(f)
      return
    else
      while True: continue
```

- What happens if we run `r(r)` ?
- If it loops, `q(r)` returns `True` and it does not loop — contradiction.
- Scooping theLoop Snooper: A proof that the Halting Problem is undecidable Geoffrey K Pullum (21 May 2024)

Why undecidable problems must exist

- A *problem* is really membership of a string in some language
- The number of different languages over any alphabet of more than one symbol is uncountable
- Programs are finite strings over a finite alphabet (ASCII or Unicode) and hence countable.
- There must be an infinity (big) of problems more than programs.
- **Computational problem** — defined by a function
- **Computational problem is computable** if there is a Turing machine that will calculate the function.

Reference: Hopcroft et al. (2007, page 318)

Computability and Terminology (1)

- The idea of an *algorithm* dates back 3000 years to Euclid, Babylonians...
- In the 1930s the idea was made more formal: which *functions* are *computable*?
- A *function* is a set of pairs $f = \{(x, f(x)) : x \in X \wedge f(x) \in Y\}$ with the *function property*
- *Function property:* $(a, b) \in f \wedge (a, c) \in f \Rightarrow b == c$
- *Function property:* Same input implies same output
- Note that maths notation is deeply inconsistent here — see Function and History of the function concept
- *What do we mean by computing a function — an algorithm ?*

Function: Relation and Rules

- The idea of function as a set of pairs (Binary relation) with the function property (each element of the domain has at most one element in the co-domain) is fairly recent — see History of the function concept
- School maths presents us with function as rule to get from the input to the output
- Example: the square function: square $x = x \times x$
- But lots of rules (or algorithms) can implement the same function
- square1 x = x^2
- square2 x = $\overbrace{x + \cdots + x}^{x\ \texttt{times}}$ if x is integer

Computability and Terminology (2)

- In the 1930s three definitions:
- λ-Calculus, simple semantics for computation — Alonzo Church
- General recursive functions — Kurt Gödel
- Universal (Turing) machine — Alan Turing

- Terminology:
  - Recursive, recursively enumerable — Church, Kleene
  - Computable, computably enumerable — Gödel, Turing
  - Decidable, semi-decidable, highly undecidable
  - In the 1930s, *computers* were *human*
  - Unfortunate choice of terminology
- Turing and Church showed that the above three were equivalent
- Church-Turing thesis — function is intuitively computable if and only if Turing machine computable

Sources on Computability Terminology

- Soare (1996) on the history of the terms *computable* and *recursive* meaning *calculable*
- See also Soare (2013, sections 9.9–9.15) in Copeland et al. (2013)

ToC

3.3.2 Reductions & Non-Computability

Reducing one problem to another

- To reduce problem P_1 to P_2, invent a construction that converts instances of P_1 to P_2 that have the same answer. That is:
  - any string in the language P_1 is converted to some string in the language P_2
  - any string over the alphabet of P_1 that is not in the language of P_1 is converted to a string that is not in the language P_2
- With this construction we can solve P_1
  - Given an instance of P_1, that is, given a string w that may be in the language P_1, apply the construction algorithm to produce a string x
  - Test whether x is in P_2 and give the same answer for w in P_1

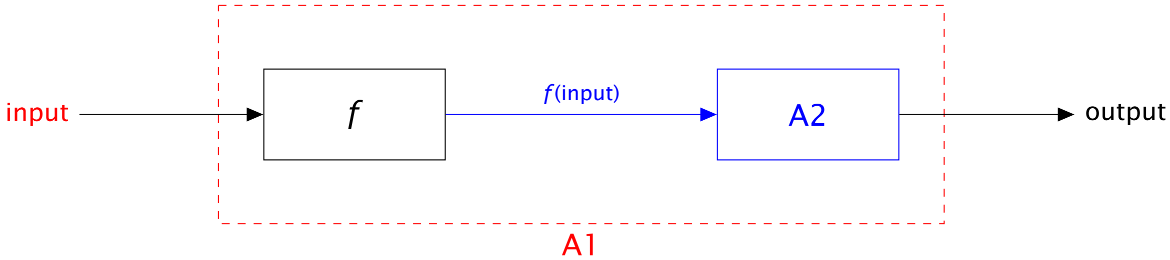
(Hopcroft et al., 2007, page 322)

- **Problem Reduction — Ordinary Example**
- Want to phone Alice but don't have her number
- You know that Bill has her number
- So *reduce* the problem of finding Alice's number to the problem of getting hold of Bill

(Rich, 2007, page 449)

- The direction of reduction is important
- If we can reduce P_1 to P_2 then (in some sense) P_2 is at least as hard as P_1 (since a solution to P_2 will give us a solution to P_1)

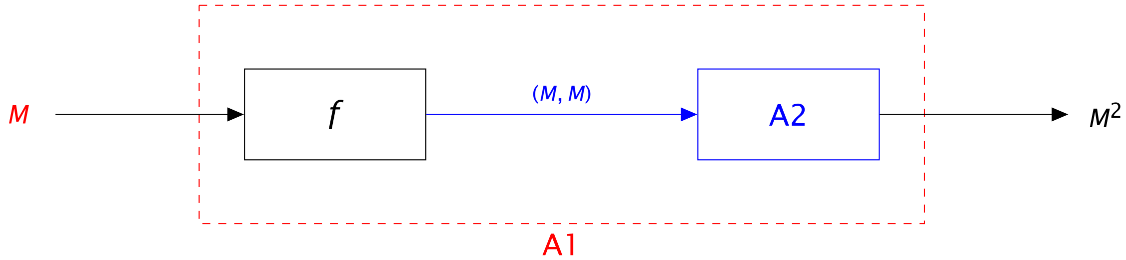
- So, if P_2 is decidable then P_1 is decidable
- To show a problem is undecidable we have to reduce from an known undecidable problem to it
- $\forall x(\text{dp}_{P_1}(x) = \text{dp}_{P_2}(\text{reduce}(x)))$
- Since, if P_1 is undecidable then P_2 is undecidable
- Some further examples
- Totality and Equivalence Problems http://www.cs.ucc.ie/~dgb/courses/toc/handout36.pdf
- Totality and Equivalence Problems https://www.cs.rochester.edu/~nelson/courses/csc_173/computability/undecidable.html — (29 April 2022) was at Undecidability
- See CS 3813 Formal Languages and Automata (26 May 2022)



- A *reduction* of problem P_1 to problem P_2
  - transforms inputs to P_1 into inputs to P_2
  - runs algorithm $A2$ (which solves P_2) and
  - interprets the outputs from $A2$ as answers to P_1
- More formally: A problem P_1 is *reducible* to a problem P_2 if there is a function f that takes any input x to P_1 and transforms it to an input $f(x)$ of P_2

  such that the solution of P_2 on $f(x)$ is the solution of P_1 on x

Source: Bridge Theory of Computation, 2007



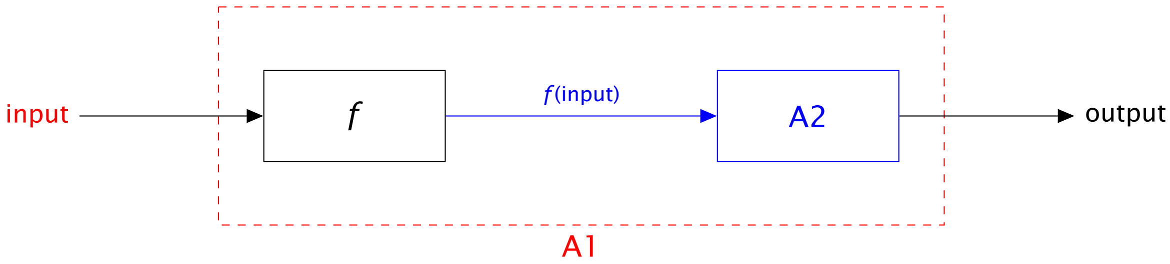
- Given an algorithm ($A2$) for matrix multiplication (P_2)
  - Input: pair of matrices, (M_1, M_2)
  - Output: matrix result of multiplying M_1 and M_2
- P_1 is the problem of squaring a matrix
  - Input: matrix M
  - Output: matrix M^2

- Algorithm $A1$ has

  $f(M) = (M, M)$

  uses $A2$ to calculate $M \times M = M^2$

Non-Computable Problems

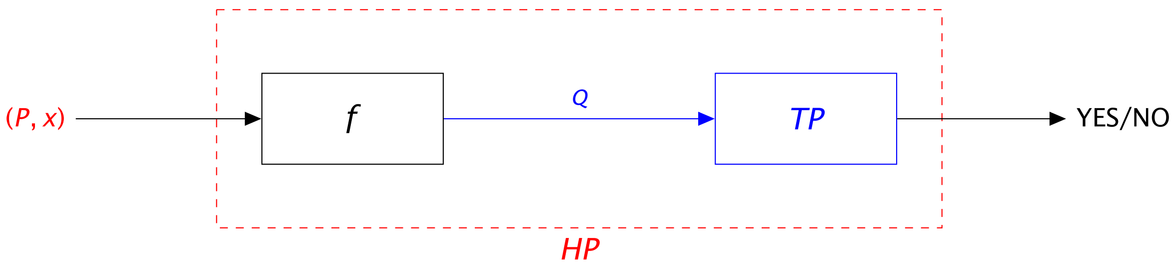


- If P_2 is computable ($A2$ exists) then P_1 is computable (f being simple or polynomial)
- Equivalently If P_1 is non-computable then P_2 is non-computable
- **Exercise:** show $B \to A \equiv \neg A \to \neg B$
- Proof by Contrapositive
- $B \to A \equiv \neg B \vee A$ by truth table or equivalences

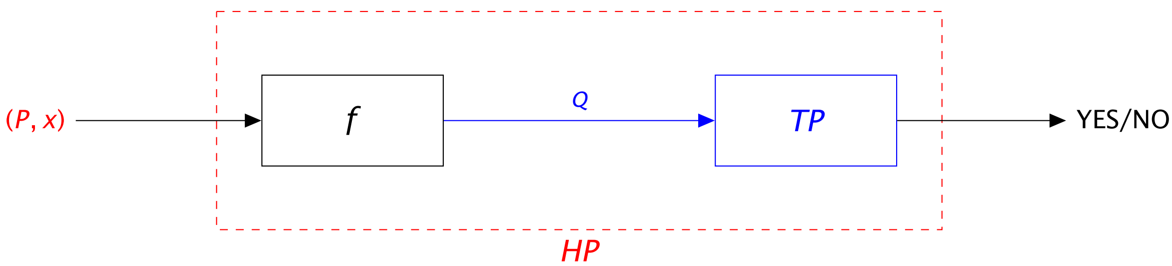
  $\equiv \neg(\neg A) \vee \neg B$ commutativity and negation laws

  $\equiv \neg A \to \neg B$ equivalences
- Common error: switching the order round

Totality Problem



- **Totality Problem**
  - Input: program Q
  - Output: YES if Q terminates for all inputs else NO
- Assume we have algorithm TP to solve the Totality Problem
- Now reduce the Halting Problem to the Totality Problem

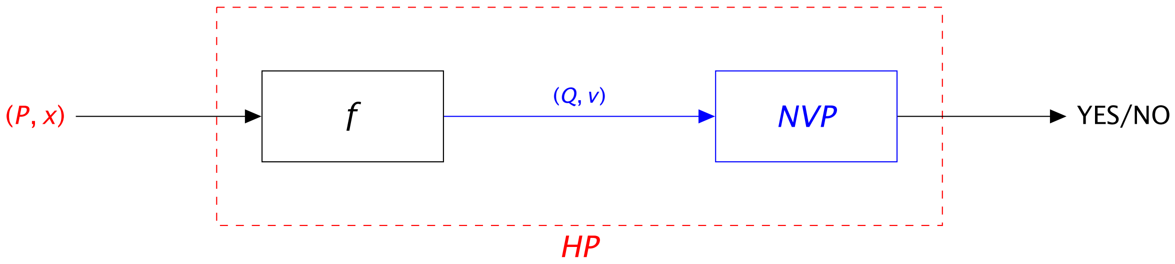


- Define f to transform inputs to HP to TP pseudo-Python

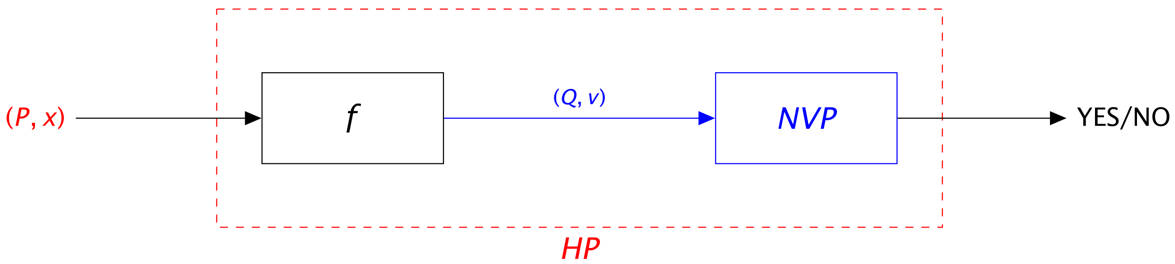
```
def f(P,x) :
  def Q(y):
    # ignore y
    P(x)
  return Q
```

- Run TP on Q
  - If TP returns YES then P halts on x
  - If TP returns NO then P does not halt on x
- We have *solved* the Halting Problem — contradiction

Negative Value Problem



- **Negative Value Problem**
  - Input: program Q which has no input and variable v used in Q
  - Output: YES if v ever gets assigned a negative value else NO
- Assume we have algorithm NVP to solve the Negative Value Problem
- Now reduce the Halting Problem to the Negative Value Problem

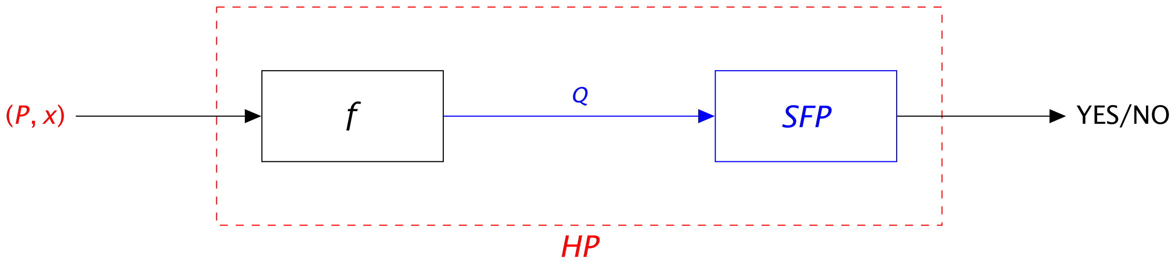


- Define f to transform inputs to HP to NVP pseudo-Python

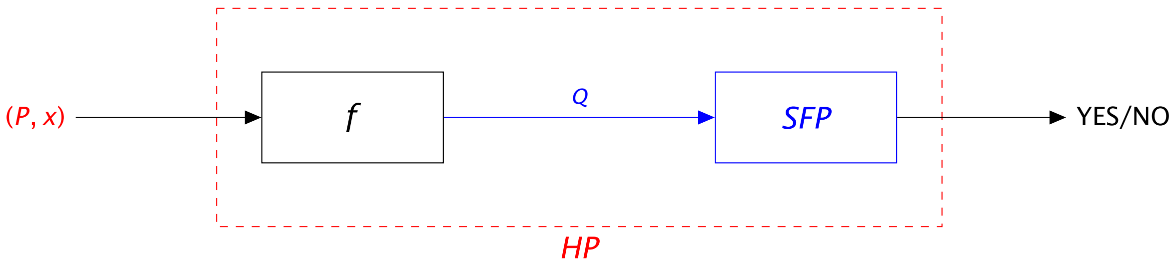
```
def f(P,x) :
  def Q(y):
    # ignore y
    P(x)
    v = -1
  return (Q,var(v))
```

- Run NVP on $(Q, var(v))$ `var(v)` gets the variable name
  - If NVP returns YES then P halts on x
  - If NVP returns NO then P does not halt on x
- We have *solved* the Halting Problem — contradiction

Squaring Function Problem



- **Squaring Function Problem**
  - Input: program *Q* which takes an integer, *y*
  - Output: YES if *Q* always returns the square of *y* else NO
- Assume we have algorithm *SFP* to solve the Squaring Function Problem
- Now reduce the Halting Problem to the Squaring Function Problem

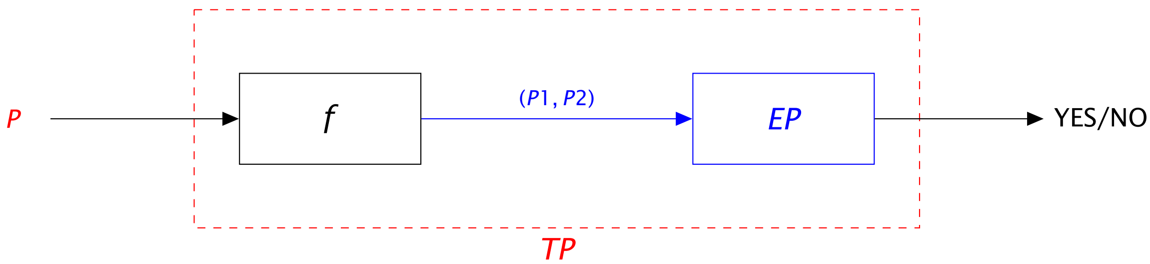


- Define *f* to transform inputs to HP to SFP pseudo-Python

```
def f(P,x) :
  def Q(y):
    P(x)
    return y * y
  return Q
```

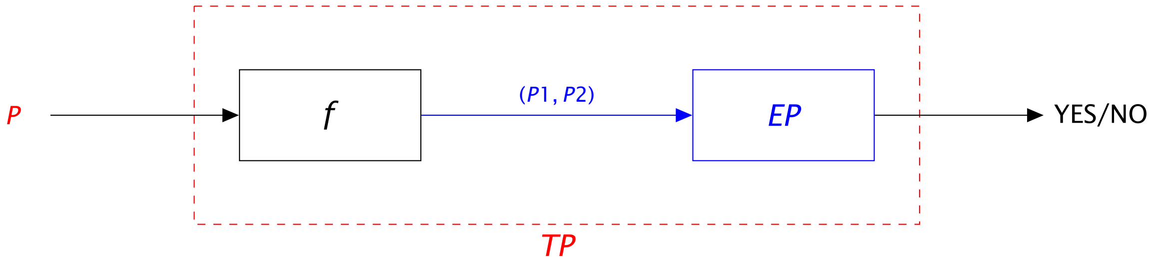
- Run *SFP* on *Q*
  - If *SFP* returns YES then *P* halts on *x*
  - If *SFP* returns NO then *P* does not halt on *x*
- We have *solved* the Halting Problem — contradiction

Equivalence Problem



- **Equivalence Problem**
  - Input: two programs *P*1 and *P*2
  - Output: YES if *P*1 and *P*2 solve the same problem (same output for same input) else NO

- Assume we have algorithm *EP* to solve the Equivalence Problem
- Now reduce the Totality Problem to the Equivalence Problem

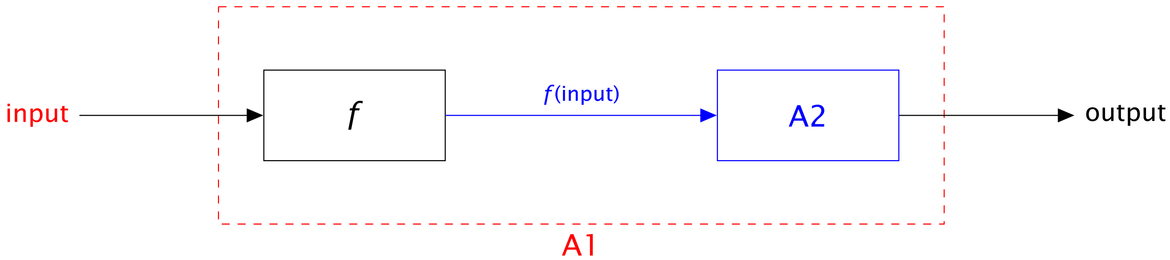


- Define *f* to transform inputs to TP to EP pseudo-Python

```
def f(P) :
  def P1(x):
    P(x)
    return "Same␣string"
  def P2(x)
    return "Same␣string"
  return (P1,P2)
```

- Run *EP* on $(P1, P2)$
  - If *EP* returns YES then *P* halts on all inputs
  - If *EP* returns NO then *P* does not halt on all inputs
- We have *solved* the Totality Problem — contradiction

Rice's Theorem



- Rice's Theorem all non-trivial, semantic properties of programs are undecidable. H G Rice 1951 PhD Thesis
- Equivalently: For any non-trivial property of partial functions, no general and effective method can decide whether an algorithm computes a partial function with that property.
- A property of partial functions is called trivial if it holds for all partial computable functions or for none.
- Rice's Theorem and computability theory
- Let S be a set of languages that is nontrivial, meaning
  - there exists a Turing machine that recognizes a language in S
  - there exists a Turing machine that recognizes a language not in S
- Then, it is undecidable to determine whether the language recognized by an arbitrary Turing machine lies in S.

- This has implications for compilers and virus checkers
- Note that Rice's theorem does not say anything about those properties of machines or programs that are not also properties of functions and languages.
- For example, whether a machine runs for more than 100 steps on some input is a decidable property, even though it is non-trivial.

ToC

3.4 Lambda Calculus

3.4.1 Motivation

- Lambda Calculus is a formal system in mathematical logic for expressing computation based on function abstraction and application using variable binding and substitution
- Lambda calculus is Turing complete — it can simulate any Turing machine
- Introduced by Alonzo Church in 1930s
- Basis of functional programming languages — Lisp, Scheme, ISWIM, ML, SASL, KRC, Miranda, Haskell, Scala, F#...
- **Note** this is not part of M269 but may help understand ideas of computability

Functions — Binding and Substitution

- School maths introduces functions as

  $f(x) = 3x^2 + 4x + 5$
- Substitution: $f(2) = 3 \times 2^2 + 4 \times 2 + 5 = 25$
- Generalise: $f(x) = ax^2 + bx + c$
- What is wrong with the following:
- $f(a) = a \times a^2 + b \times a + c$
- The ideas of free and bound variables and substitution

Expressions — Evaluation Strategies

- In evaluating an expression we have choices about the order in which we evaluate subterms
- Some choices may involve more work than others but the Church-Rosser theorem ensures that if the evaluation terminates then all choices get to the same answer
- The second edition of a famous book on Functional programming — Bird (1998, Ex 1.2.2, page 6) — had the following exercise:
- How many ways can you evaluate $(3 + 7)^2$

  List the evaluations and assumptions
- The first edition — Bird and Wadler (1988, Ex 1.2.2, page 6) — had the exercise:

- How many ways can you evaluate $\left((3+7)^2\right)^2$
- How many ways can you evaluate $(3+7)^2$

  List the evaluations and assumptions
- **Answer** 3 ways
- **Reducible expressions** (redexes)

  $x^2 \rightarrow x \times x$ where x is a term

  $a + b$ where a and b are numbers

  $x \times y$ where x and y are numbers

```
1 [sqr (3+7),((3+7)*(3+7)),((3+7)*10),(10*10),100]
2 [sqr (3+7),((3+7)*(3+7)),(10*(3+7)),(10*10),100]
3 [sqr (3+7),sqr 10,(10*10),100]
```

- The assumed redexes do not include distributive laws

  $(a+b) \times (x+y) \rightarrow a \times x + a \times y + b \times x + b \times y$
- This would increase the number of different evaluations
- How many ways can you evaluate $\left((3+7)^2\right)^2$
- **Answer** 547 ways

```
1 [sqr sqr (3+7),(sqr (3+7)*sqr (3+7)),(sqr (3+7)*((3+7)*(3+7))), (sqr (3+7)*((3+7)*10)),(sqr (3+7)*(10*10)), (sqr (3+7)*100),(((3+7)*(3+7))*100),(((3+7)*10)*100), ((10*10)*100),(100*100),10000]
2 [sqr sqr (3+7),(sqr (3+7)*sqr (3+7)),(sqr(3+7)*((3+7)*(3+7))), (sqr (3+7)*((3+7)*10)),(sqr (3+7)*(10*10)), (sqr (3+7)*100),(((3+7)*(3+7))*100),((10*(3+7))*100), ((10*10)*100),(100*100),10000]
```

```
546 [sqr sqr (3+7),sqr sqr 10,sqr (10*10), ((10*10)*(10*10)),(100*(10*10)),(100*100),10000]
547 [sqr sqr (3+7),sqr sqr 10,sqr (10*10),sqr 100,(100*100),10000]
```

- Enumerating all 547 ways may have taken some concentration
- The actual Evaluation strategy used by a particular programming language implementation may have optimisations which make an evaluation which looks costly to be somewhat cheaper
- For example, the Haskell implementation GHC optimises the evaluation of common subexpressions so that `(3+7)` will be evaluated only once

```
1 [sqr sqr (3+7),(sqr (3+7)*sqr (3+7)),(sqr (3+7)*((3+7)*(3+7))), (sqr (3+7)*((3+7)*10)),(sqr (3+7)*(10*10)), (sqr (3+7)*100),(((3+7)*(3+7))*100),(((3+7)*10)*100), ((10*10)*100),(100*100),10000]
2 [sqr sqr (3+7),(sqr (3+7)*sqr (3+7)),(sqr(3+7)*((3+7)*(3+7))), (sqr (3+7)*((3+7)*10)),(sqr (3+7)*(10*10)), (sqr (3+7)*100),(((3+7)*(3+7))*100),((10*(3+7))*100), ((10*10)*100),(100*100),10000]
```

- M269 Unit 6/7 Reader *Logic and the Limits of Computation* alludes to other formalisations with equal power to a Turing Machine (pages 81 and 87)
- The *Reader* mentions Alonzo Church and his 1930s formalism (page 87, but does not give any detail)
- The notes in this section are optional and for comparison with the Turing Machine material

- Turing machine: explicit memory, state and implicit loop and case/if statement
- Lambda Calculus: function definition and application, explicit rules for evaluation (and transformation) of expressions, explicit rules for substitution (for function application)
- Lambda calculus reduction workbench
- Lambda Calculus Calculator

ToC

3.4.2 Lambda Terms

- A **variable**, x, is a lambda term
- If M is a lambda term and x is a variable, then $(\lambda x.M)$ is a lambda term — a **lambda abstraction** or function definition
- If M and N are lambda terms, the $(M\ N)$ is lambda term — an **application**
- Nothing else is a lambda term

(Lambda Calculus notes based on lecture slides at CMSC 330, Spring 2011)

- Outermost parentheses are omitted $(M\ N) \equiv M\ N$
- Application is left associative $((M\ N)\ P) \equiv M\ N\ P$
- The body of an abstraction extends as far right as possible, subject to scope limited by parentheses
- $\lambda x.M\ N \equiv \lambda x.(M\ N)$ and not $(\lambda x.M)\ N$
- $\lambda x.\lambda y.\lambda z.M \equiv \lambda x\,y\,z.M$

Lambda Calculus Semantics

- What do we mean by *evaluating an expression*?
- To evaluate $(\lambda x.M)N$
- Evaluate M with x replaced by N
- This rule is called β-reduction
- $(\lambda x.M)N \underset{\beta}{\rightarrow} M[x := N]$
- $M[x := N]$ is M with occurrences of x replaced by N
- This operation is called *substitution* — see rules below

β-Reduction Examples

- $(\lambda x.x)z \rightarrow z$
- $(\lambda x.y)z \rightarrow y$
- $(\lambda x.x\,y)z \rightarrow z\,y$

  a function that applies its argument to y

- $(\lambda x.x\, y)(\lambda z.z) \rightarrow (\lambda z.z)y \rightarrow y$
- $(\lambda x.\lambda y.x\, y)z \rightarrow \lambda y.z\, y$

  A *curried* function of two arguments — applies first argument to second
- *currying* replaces $f(x, y)$ with $(f\, x)y$ — nice notational convenience — gives *partial application* for free

ToC

3.4.3 Substitution

- To define *substitution* use recursion on the structure of terms
- $x[x := N] \equiv N$
- $y[x := N] \equiv y$
- $(P\, Q)[x := N] \equiv (P[x := N])\,(Q[x := N])$
- $(\lambda x.M)[x := N] = \lambda x.M$

  In $(\lambda x.M)$, the x is a formal parameter and thus a local variable, different to any other
- $(\lambda y.M)[x := N] =$ what?
- Look back at the school maths example above — a subtle point
- Renaming *bound variables consistently is allowed*
- $\lambda x.x \equiv \lambda y.y \equiv \lambda z.z$
- $\lambda y.\lambda x.y \equiv \lambda z.\lambda x.z$
- This is called α*-conversion*
- $(\lambda x.\lambda y.x\, y)\, y \rightarrow (\lambda x.\lambda z.x\, z)\, y \rightarrow \lambda z.y\, z$
- **Bound and Free Variables**
- $BV(x) = \emptyset$
- $BV(\lambda x.M) = BV(M) \cup \{x\}$
- $BV(M\, N) = BV(M) \cup BV(N)$
- $FV(x) = \{x\}$
- $FV(\lambda x.M) = FV(M) - \{x\}$
- $FV(M\, N) = FV(M) \cup FV(N)$
- The above is a formalisation of school maths
- A Lambda term with no free variables is said to be *closed* — such terms are also called **combinators** — see Combinator and Combinatory logic (Hankin, 2004, page 10)
- **α-conversion**
- $\lambda x.M \underset{\alpha}{\rightarrow} \lambda y.M[x := y]$ if $y \notin FV(M)$
- β-reduction final rule

- $(\lambda y.M)[x := N] = \lambda y.M$ if $x \notin FV(M)$
- $(\lambda y.M)[x := N] = \lambda y.M[x := N]$

  if $x \in FV(M)$ and $y \notin FV(N)$
- $(\lambda y.M)[x := N] = \lambda z.M[y := z][x := N]$

  if $x \in FV(M)$ and $y \in FV(N)$

  z is chosen to be first variable $z \notin FV(N\,M)$
- This is why you cannot go $f(a)$ when given
- $f(x) = ax^2 + bx + c$
- School maths — but made formal

Lambda Calculus — Rules Summary — Conversion

- α-**conversion** renaming bound variables
- $\lambda x.M \underset{\alpha}{\rightarrow} \lambda y.M[x := y]$ if $y \notin FV(M)$
- β-**conversion** function application
- $(\lambda x.M)N \underset{\beta}{\rightarrow} M[x := N]$
- η-**conversion** extensionality
- $\lambda x.F\,x \underset{\eta}{\rightarrow} F$ if $x \notin FV(F)$

Lambda Calculus — Rules Summary — Substitution

1. $x[x := N] \equiv N$
2. $y[x := N] \equiv y$
3. $(P\,Q)[x := N] \equiv (P[x := N])\,(Q[x := N])$
4. $(\lambda x.M)[x := N] = \lambda x.M$
5. $(\lambda y.M)[x := N] = \lambda y.M$ if $x \notin FV(M)$
6. $(\lambda y.M)[x := N] = \lambda y.M[x := N]$

   if $x \in FV(M)$ and $y \notin FV(N)$
7. $(\lambda y.M)[x := N] = \lambda z.M[y := z][x := N]$

   if $x \in FV(M)$ and $y \in FV(N)$

   z is chosen to be first variable $z \notin FV(N\,M)$

ToC

3.4.4 Lambda Calculus Encodings

- So what does this formalism get us ?
- The Lambda Calculus is Turing complete

- We can encode any computation (if we are clever enough)
- Booleans and propositional logic
- Pairs
- Natural numbers and arithmetic
- Looping and recursion

Booleans and Propositional Logic

- True $= \lambda x.\lambda y.x$
- False $= \lambda x.\lambda y.y$
- IF a THEN b ELSE $c \equiv a\,b\,c$
- IF True THEN b ELSE $c \to (\lambda x.\lambda y.x)\;b\;c$
- $\quad\to (\lambda y.b)\;c \to b$
- IF False THEN b ELSE $c \to (\lambda x.\lambda y.y)\;b\;c$
- $\quad\to (\lambda y.y)\;c \to c$
- Not $= \lambda x.((x\,\text{False})\text{True})$
- Not x = IF x THEN False ELSE True
- Exercise: evaluate Not True
- And $= \lambda x.\lambda y.((x\;y)\;\text{False})$
- And x y = IF x THEN y ELSE False
- Exercise: evaluate And True False
- Or $= \lambda x.\lambda y.((x\;\text{True}\;)\;y)$
- Or x y = IF x THEN True ELSE y
- Exercise: evaluate Or False True
- Exercise: evaluate Not True
- $\to (\lambda x.((x\;\text{False})\;\text{True}))\;\;\text{True}$
- $\to$ (True False) True
- Could go straight to False from here, but we shall fill in the detail
- $\to ((\lambda x.\lambda y.x)\,(\lambda x.\lambda y.y))\,(\lambda x.\lambda y.x)$
- $\to (\lambda y.(\lambda x.\lambda y.y))\,(\lambda x.\lambda y.x)$
- $\to (\lambda x.\lambda y.y) \equiv$ False
- Exercise: evaluate And True False
- $\to$(IF x THEN y ELSE False) True False
- $\to$(IF True THEN False ELSE False) $\to$False
- Exercise: evaluate Or False True

- $\rightarrow$(IF x THEN True ELSE y) False True
- $\rightarrow$(IF False THEN True ELSE True) $\rightarrow$True

Natural Numbers — Church Numerals

- Encoding of natural numbers
- $0 = \lambda f.\lambda y.y$
- $1 = \lambda f.\lambda y.f\ y$
- $2 = \lambda f.\lambda y.f\ (f\ y)$
- $3 = \lambda f.\lambda y.f\ (f\ (f\ y))$
- Successor $\text{Succ} = \lambda z.\lambda f.\lambda y.f(z\ f\ y)$
- $\text{Succ}\ 0 = (\lambda z.\lambda f.\lambda y.f(z\ f\ y))(\lambda f.\lambda y.y)$
- $\rightarrow \lambda f.\lambda y.f\ ((\lambda f.\lambda y.y)\ f\ y)$
- $\rightarrow \lambda f.\lambda y.f\ ((\lambda y.y)\ y)$
- $\rightarrow \lambda f.\lambda y.f\ y = 1$

Natural Numbers — Operations

- $\text{isZero} = \lambda z.z(\lambda y.\ \text{False}\ )\ \text{True}$
- Exercise: evaluate isZero 0
- If M and N are numerals (as λ expressions)
- Add $M\,N = \lambda x.\lambda y.(M\,x)\,((N\,x)\,y)$
- Mult $M\,N = \lambda x.(M\,(N\,x))$
- Exercise: show $1 + 1 = 2$

Pairs

- Encoding of a pair a, b
- $(a, b) = \lambda x.$ IF x THEN a ELSE b
- $\text{FST} = \lambda f.f\ \text{True}$
- $\text{SND} = \lambda f.f\ \text{False}$
- Exercise: evaluate FST (a, b)
- Exercise: evaluate SND (a, b)

The Fixpoint Combinator

- $Y = \lambda f.(\lambda x.f\,(x\,x))\,(\lambda x.f\,(x\,x))$
- $Y\,F = \lambda f.(\lambda x.f\,(x\,x))\,(\lambda x.f\,(x\,x))\,F$
- $\rightarrow (\lambda x.F\,(x\,x))(\lambda x.F\,(x\,x))$
- $F((\lambda x.F\,(x\,x))\,(\lambda x.F\,(x\,x))) = F\,(Y\,F)$

- $(Y\ F)$ is a fixed point of F
- We can use Y to achieve recursion for F
- **Recursion implementation — Factorial**
- Fact $= \lambda f.\lambda n.$ IF $n = 0$ THEN 1 ELSE $n * (f\,(n - 1))$
- $(Y$ Fact$)1 =$ (Fact $(Y$ Fact$))1$
- $\rightarrow$ IF $1 = 0$ THEN 1 ELSE $1 * ((Y$ Fact$)\,0)$
- $\rightarrow 1 * ((Y$ Fact$)\,0)$
- $\rightarrow 1 *$ (Fact $(Y$ Fact$)\,0)$
- $\rightarrow 1 *$ IF $0 = 0$ THEN 1 ELSE $0 * ((Y$ Fact$)\,(0 - 1))$
- $\rightarrow 1 * 1 \rightarrow 1$
- Factorial n = (Y Fact) n
- Recursion implemented with a non-recursive function Y

ToC

Turing Machines, Lambda Calculus and Programming Languages

- Anything computable can be represented as TM or Lambda Calculus
- But programs would be slow, large and hard to read
- In practice use the ideas to create more expressive languages which include built-in primitives
- Also leads to ideas on data types
- Polymorphic data types
- Algebraic data types
- Also leads on to ideas on higher order functions — functions that take functions as arguments or returns functions as results.

ToC

Commentary 3

3 Complexity

- Complexity Classes **P** and **NP**
- Class **NP**
- NP-completeness
- NP-completeness and Boolean Satisfiability

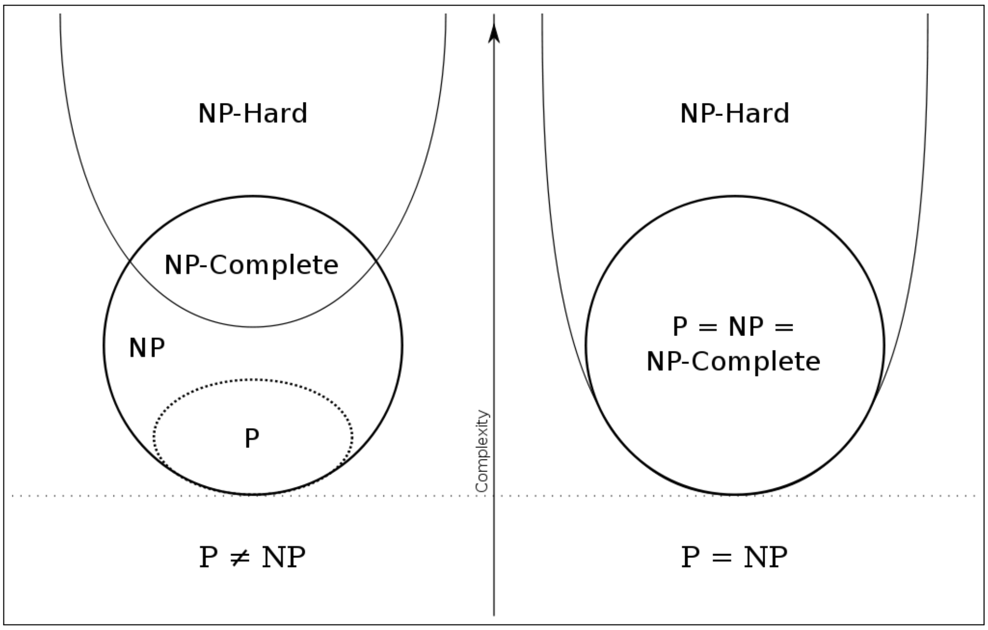
ToC

4 Complexity

4.1 Complexity Classes P and NP

- P, the set of all decision problems that can be solved in polynomial time on a deterministic Turing machine
- NP, the set of all decision problems whose solutions can be verified (certificate) in polynomial time
- Equivalently, NP, the set of all decision problems that can be solved in polynomial time on a non-deterministic Turing machine
- A *decision problem*, *dp* is *NP-complete* if
  1. *dp* is in NP and
  2. Every problem in NP is reducible to *dp* in polynomial time
- NP-hard — a problem satisfying the second condition, whether or not it satisfies the first condition. Class of problems which are at least as hard as the hardest problems in NP. NP-hard problems do not have to be in NP and may not be decision problems

Euler diagram for P, NP, NP-complete and NP-hard set of problems



Source: Wikipedia NP-complete entry

ToC

4.2 Class NP

- To formalise the definition of the class **NP**, we need to formalise the idea of checking a candidate solution
- Define a *certificate* for each problem input that would return Yes
- Describe the *verifier* algorithm
- Demonstrate the *verifier* algorithm has polynomial complexity

- The terms *certificate* and *verifier* have technical definitions in terms of languages and Turing Machines but can be thought of as *candidate solution* and *checker algorithm*

Example NP Decision Problems

- **Composite Numbers** Given a number N decide if N is a composite (i.e. non-prime) number

  Certificate factorization of N

- **Connectivity** Given a graph G and two vertices s, t in G, decide if s is connected to t in G.

  Certificate path from s to t

- **Linear Programming** Given a list of m linear inequalities with rational coefficients over n variables $u_1, \ldots, u_n$ (a linear inequality has the form $a_1 u_1 + a_2 u_2 \cdots + a_n u_n \leqslant b$ for some coefficients $a_1, \ldots, a_n b$), decide if there is an assignment of rational numbers to the variables $u_1, \ldots, u_n$ which satisfies all the inequalities

  Certificate is the assignment

- The above are in **P**
- *Composite Numbers*, *Connectivity* and *Linear programming* are in **P**
- *Composite Numbers* follows from Integer factorization and the AKS primality test from 2004
- *Connectivity* follows from the breadth-first search algorithm
- *Linear programming* shown to be in **P** by the Ellipsoid method
- **Integer Programming** some or all variables are restricted to be integers
- **Travelling Salesperson** Given a set of nodes and distances between all pairs of nodes and a number k, decide if there is a closed circuit that visits every node exactly once and has total length at most k

  Certificate sequence of nodesin such a tour

- **Subset sum** Given a list of numbers and a number T, decide if there is a subset that adds up to T

  Certificate list of members of such a subset

- **Independent set (graph theory)** A subgraph of G with of at least k vertices which have no edges between them

  Certificate the list of k vertices

- **Clique problem** Given a graph and a number k, decide if there is a complete subgraph (clique) of size k

  Certificate list pf nodes. For explanation see Prove Clique is NP

- The above are **NP-complete** — see List of NP-complete problems
- The following two are not known to be **P** nor **NP-complete**

- **Graph Isomorphism** Given two $n \times n$ adjacency matrices M_1, M_2, decide if M_1 and M_2 define the same graph (up to renaming of the vertices)

  Certificate the permutation $\pi : [n] \to [n]$ such that M_2 is equal to M_1 after reordering the indices of M_1 according to π

- **Integer factorization** Given three numbers N, L, U decide if N has a prime factor p in the interval $[L, U]$

  Certificate is the factorization of N

  Source Arora and Barak (2009, page 49) *Computational Complexity: A Modern Approach* and contained links

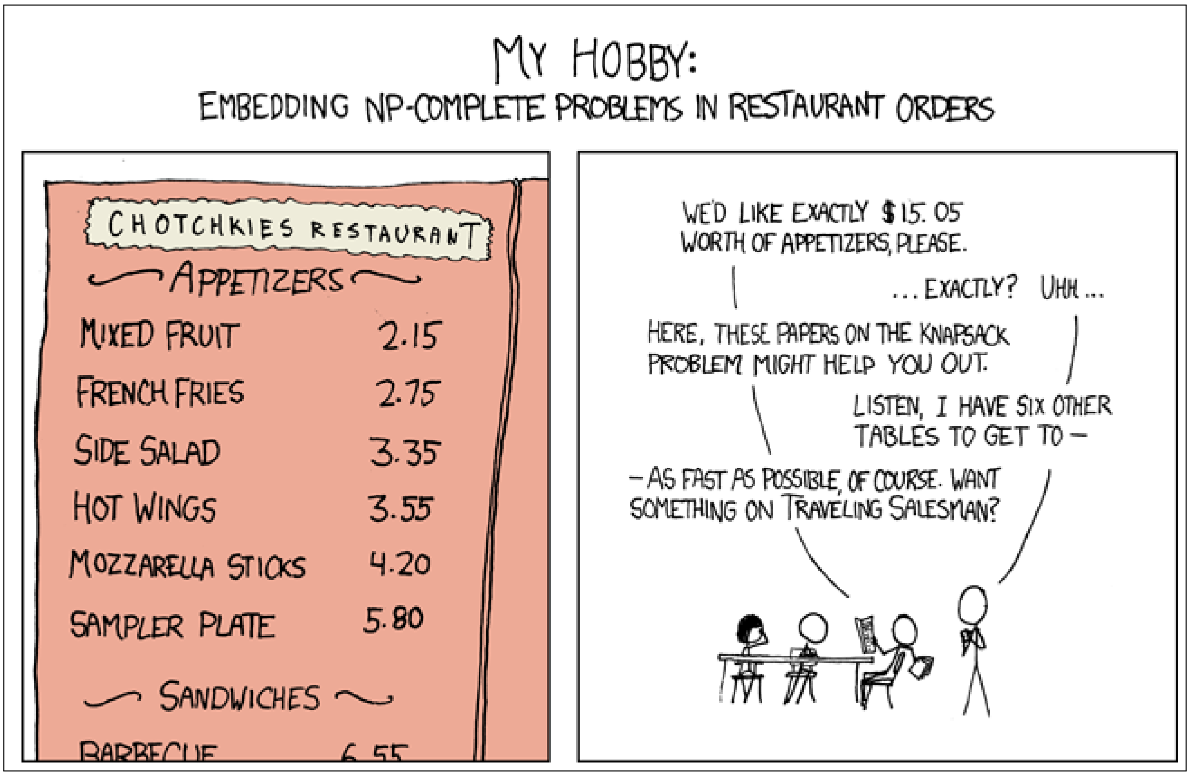
ToC

4.3 NP-completeness

NP-complete problems

- Boolean satisfiability (SAT) Cook-Levin theorem
- Conjunctive Normal Form 3SAT
- Hamiltonian path problem
- Travelling salesman problem
- NP-complete — see list of problems

XKCD on NP-Complete Problems



Source & Explanation: XKCD 287

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4.4 NP-Completeness and Boolean Satisfiability

- The *Boolean satisfiability problem (SAT)* was the first decision problem shown to be *NP-Complete*
- This section gives a sketch of an explanation
- **Health Warning** different texts have different notations and there will be some inconsistency in these notes
- **Health warning** these notes use some formal notation *to make the ideas more precise* — computation requires precise notation and is about manipulating strings according to precise rules.

Alphabets, Strings and Languages

- Notation:
- Σ is a set of symbols — the alphabet
- Σ^k is the set of all string of length k, which each symbol from Σ
- Example: if $\Sigma = \{0, 1\}$
  - $\Sigma^1 = \{0, 1\}$
  - $\Sigma^2 = \{00, 01, 10, 11\}$
- $\Sigma^0 = \{\epsilon\}$ where ϵ is the empty string
- Σ^* is the set of all possible strings over Σ
- $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$
- A *Language*, L, over Σ is a subset of Σ^*
- $L \subseteq \Sigma^*$

Language Accepted by a Turing Machine

- Language accepted by Turing Machine, M denoted by $L(M)$
- $L(M)$ is the set of strings $w \in \Sigma^*$ accepted by M
- For *Final States* $F = \{Y, N\}$, a string $w \in \Sigma^*$ is accepted by $M \Leftrightarrow$ (if and only if) M starting in q_0 with w on the tape halts in state Y
- Calculating a function (function problem) can be turned into a decision problem by asking whether $f(x) = y$

The NP-Complete Class

- If we do not know if P $\neq$ NP, what can we say ?
- A language L is *NP-Complete* if:
  - $L \in$ NP and
  - for all other $L' \in$ NP there is a *polynomial time transformation* (Karp reducible, reduction) from L' to L

- Problem P_1 *polynomially reduces* (Karp reduces, transforms) to P_2, written $P_1 \propto P_2$ or $P_1 \leq_p P_2$, iff $\exists f : \mathrm{dp}_{P_1} \to \mathrm{dp}_{P_2}$ such that
  - $\forall I \in \mathrm{dp}_{P_1} [I \in Y_{P_1} \Leftrightarrow f(I) \in Y_{P_2}]$
  - f can be computed in polynomial time
- More formally, $L_1 \subseteq \Sigma_1^*$ polynomially transforms to $L_2 \subseteq \Sigma_2^*$, written $L_1 \propto L_2$ or $L_1 \leq_p L_2$, iff $\exists f : \Sigma_1^* \to \Sigma_2^*$ such that
  - $\forall x \in \Sigma_1^* [x \in L_1 \Leftrightarrow f(x) \in L_2]$
  - There is a polynomial time TM that computes f
- *Transitivity* If $L_1 \propto L_2$ and $L_2 \propto L_3$ then $L_1 \propto L_3$
- If L is NP-Hard and $L \in$ P then P = NP
- If L is NP-Complete, then $L \in$ P if and only if P = NP
- If L_0 is NP-Complete and $L \in$ NP and $L_0 \propto L$ then L is NP-Complete
- Hence if we find one NP-Complete problem, it may become easier to find more
- In 1971/1973 Cook-Levin showed that the Boolean satisfiability problem (SAT) is NP-Complete

The Boolean Satisfiability Problem

- A propositional logic formula or Boolean expression is built from variables, operators: AND (conjunction, $\wedge$), OR (disjunction, $\vee$), NOT (negation, $\neg$)
- A formula is said to be *satisfiable* if it can be made True by some assignment to its variables.
- *The Boolean Satisfiability Problem* is, given a formula, check if it is satisfiable.
  - *Instance:* a finite set U of Boolean variables and a finite set C of clauses over U
  - *Question:* Is there a satisfying truth assignment for C ?
- A *clause* is is a disjunction of variables or negations of variables
- *Conjunctive normal form (CNF)* is a conjunction of clauses
- Any Boolean expression can be transformed to CNF
- Given a set of Boolean variable $U = \{u_1, u_2, \ldots, u_n\}$
- A literal from U is either any u_i or the negation of some u_i (written $\overline{u_i}$) usual notation $\neg u_i$
- A clause is denoted as a subset of literals from U — $\{u_2, \overline{u_4}, u_5\}$ usual notation $u_2 \vee \neg u_4 \vee u_5$
- A clause is satisfied by an assignment to the variables if at least one of the literals evaluates to True (just like disjunction of the literals)
- Let C be a set of clauses over U — C is satisfiable iff there is some assignment of truth values to the variables so that every clause is satisfied (just like CNF)
- $C = \{\{u_1, u_2, u_3\}, \{\overline{u_2}, \overline{u_3}\}, \{u_2, \overline{u_3}\}\}$ is satisfiable

  usual notation $(u_1 \vee u_2 \vee u_3) \wedge (\neg u_2 \vee \neg u_3) \wedge (u_2 \vee \neg u_3)$

  assign $(u_1, u_2, u_3) = (T, F, F), (T, T, F), (F, T, F)$

- $C = \{\{u_1, u_2\}, \{u_1, \overline{u_2}\}, \{\overline{u_1}\}\}$ is not satisfiable

  usual notation $(u_1 \lor u_2) \land (u_1 \lor \neg u_2) \land (\neg u_1)$

- Proof that SAT is NP-Complete looks at the structure of NDTMs and shows you can transform any NDTM to SAT in polynomial time (in fact logarithmic space suffices)
- SAT is in NP since you can check a solution in polynomial time
- To show that $\forall L \in \text{NP} : L \propto \text{SAT}$ invent a polynomial time algorithm for each polynomial time NDTM, M, which takes as input a string x and produces a Boolean formula E_x which is satisfiable iff M accepts x
- See Cook-Levin theorem

Sources

- Garey and Johnson (1979, page 34) has the notation $L_1 \propto L_2$ for polynomial transformation
- Arora and Barak (2009, page 42) has the notation $L_1 \leq_p L_2$ for *polynomial-time Karp reducible*
- The sketch of Cook's theorem is from Garey and Johnson (1979, page 38)
- For the satisfiable C we could have assignments $(u_1, u_2, u_3) \in \{(T, T, F), (T, F, F), (F, T, F)\}$

Coping with NP-Completeness

- What does it mean if a problem is NP-Complete ?
  - There is a P time verification algorithm.
  - There is a P time algorithm to solve it iff P = NP (?)
  - No one has yet found a P time algorithm to solve any NP-Complete problem
  - So what do we do ?
- Improved exhaustive search — Dynamic Programming; Branch and Bound
- Heuristic methods — *acceptable* solutions in *acceptable* time — compromise on optimality
- Average time analysis — look for an algorithm with good average time — compromise on generality (see Big-O Algorithm Complexity Cheatsheet)
- Probabilistic or Randomized algorithms — compromise on correctness

Sources

- *Practical Solutions for Hard Problems* Rich (2007, chp 30)
- *Coping with NP-Complete Problems* Garey and Johnson (1979, chp 6)

ToC

5 Turing Machine TMA Question

- The transition function is represented as a Python dictionary mapping stete, symbol to symbol, move, state
- States are represented as strings — we may define Python constants to make life easier (see below)
- What are the states ?
- Tape represented by a list; moves by 1, -1, 0

```
# Moves
RIGHT = 1
LEFT  = -1
STAY  = 0

# States
Start   = "start"
FindA   = "FindA"
Find0   = "Find0"
FindNum = "FindNum"
FinishOK    = "FinishOK"
FinishNotOK = "FinishNotOK"
Stop        = "stop"
```

- Note that the identifiers must be valid Python
- Python has conventions about constantss
- Describe the actions for each state — possibly using Python dictionary notation (to make shorter work)

```
  (Start, "a"):   ("a", RIGHT,FindA),
  (Start, "0"):   ("0", RIGHT,Find0),
  (Start, "#"):   ("#", RIGHT,FindNum),
  (Start, None):  (None,STAY, Stop), # Is empty input allowed ?

  (FindA, "a"):   ("a", RIGHT,FinishOK),
  (FindA, "0"):   ("0", RIGHT,FindA),
  (FindA, "#"):   ("#", RIGHT,FindA),
  (FindA, None):  (False,STAY, Stop),

  (Find0, "a"):   ("a", RIGHT,Find0),
  (Find0, "0"):   ("0", RIGHT,FinishOK),
  (Find0, "#"):   ("#", RIGHT,Find0),
  (Find0, None):  (False,STAY, Stop),

  (FindNum, "a"):   ("a", RIGHT,FindNum),
  (FindNum, "0"):   ("0", RIGHT,FindNum),
  (FindNum, "#"):   ("#", RIGHT,FinishOK),
  (FindNum, None):  (False,STAY, Stop),
```

- FinishOK and FinishNotOK should tidy up the output and move the read/write head to an apprpropriate position

```
(FinishOK, "a"): ("a",RIGHT,FinishOK),
(FinishOK, "0"): ("0",RIGHT,FinishOK),
(FinishOK, "#"): ("#",RIGHT,FinishOK),
(FinishOK, None): (True,STAY,Stop),
```

- What if we wanted to erase everything else and only have False/True as output ?

6 Complexity and Big O Notation

- Measuring program complexity introduced in section 4 of M269 Unit 2

- See also Miller and Ranum chapter 2 Big-O Notation
- See also Wikipedia: Big O notation
- See also Big-O Cheat Sheet
- Complexity of algorithm measured by using some surrogate to get rough idea
- In M269 mainly using assignment statements
- For *exact* measure we would have to have cost of each operation, knowledge of the implementation of the programming language and the operating system it runs under.
- But mainly interested in the following questions:
- **(1)** Is algorithm A more efficient than algorithm B for large inputs ?
- **(2)** Is there a lower bound on any possible algorithm for calculating this particular function ?
- **(3)** Is it always possible to find a polynomial time (n^k) algorithm for any function that is computable
- — the later questions are addressed in Unit 7

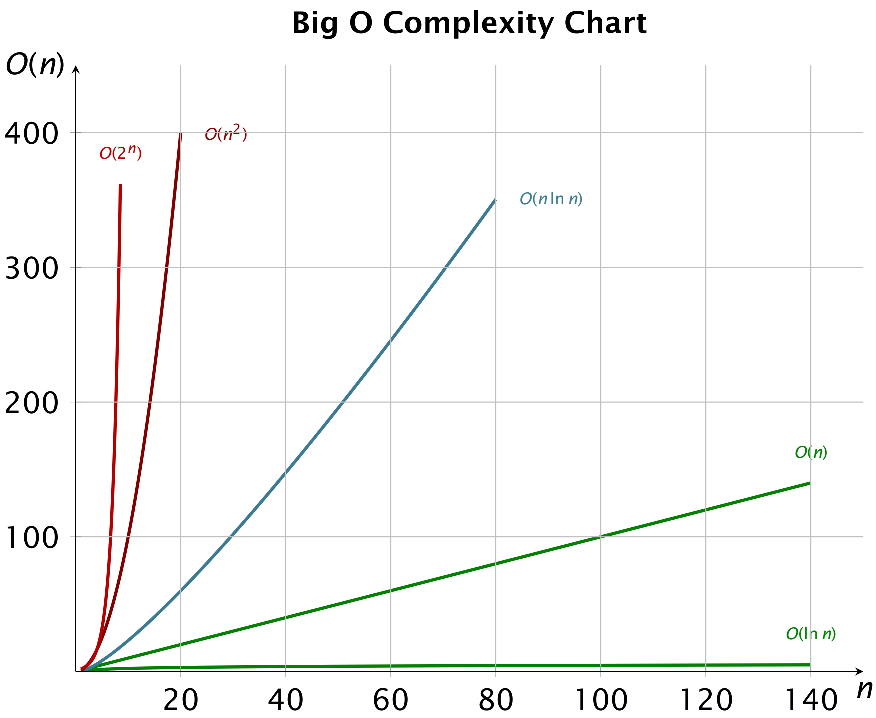
Orders of Common Functions

- $O(1)$ **constant** — look-up table
- $O(\log n)$ **logarithmic** — binary search of sorted array, binary search tree, binomial heap operations
- $O(n)$ **linear** — searching an unsorted list
- $O(n \log n)$ **loglinear** — heapsort, quicksort (best and average), merge sort
- $O(n^2)$ **quadratic** — bubble sort (worst case or naive implementation), Shell sort, quicksort (worst case), selection sort, insertion sort
- $O(n^c)$ **polynomial**
- $O(c^n)$ **exponential** — travelling salesman problem via dynamic programming, determining if two logical statements are equivalent by brute force
- $O(n!)$ **factorial** — TSP via brute force.

Tyranny of Asymptotics

- Table from Bentley (1984, page 868)
- Cubic algorithm on Cray-1 supercomputer with FORTRAN$3.0n^3$ nanoseconds
- Linear algorithm on TRS-80 micro with BASIC $19.5n \times 10^6$ nanoseconds

N	Cray-1	TRS-80
10	3.0 microsecs	200 millisecs
100	3.0 millisecs	2.0 secs
1000	3.0 secs	20 secs
10000	49 mins	3.2 mins
100000	35 days	32 mins
1000000	95 yrs	5.4 hrs



Big O Notation

- Abuse of notation — we write $f(x) = O(g(x))$
- but $O(g(x))$ is the class of all functions $h(x)$ such that $|h(x)| \leqslant C|g(x)|$ for some constant C
- So we should write $f(x) \in O(g(x))$ (but we don't)
- We ought to use a notation that says that (informally) the function f is bounded both above and below by g asymptotically
- This would mean that for big enough x we have

  $k_1 g(x) \leqslant f(x) \leqslant k_2 g(x)$ for some k_1, k_2
- This is Big Theta, $f(x) = \Theta(g(x))$
- But we use Big O to indicate an asymptotically tight bound where Big Theta might be more appropriate
- See Wikipedia: Big O Notation
- This could be Maths phobia generated confusion

6.1 Complexity Example

```python
def someFunction(aList) :
  n = len(aList)
  best = 0
  for i in range(n) :
    for j in range(i + 1, n + 1) :
      s = sum(aList[i:j])
      best = max(best, s)
  return best
```

- Example from M269 Unit 2 page 46
- Code in file M269TutorialProgPythonADT.py
- What does the code do ?
- (It was a *famous* problem from the late 1970s/early 1980s)
- Can we construct a more efficient algorithm for the same computational problem ?
- The code calculates the maximum subsegment of a list
- Described in Bentley (1984),Bentley (1986, column 7),Bentley (2000, column 8) Also in Gries (1989)
- These are all in a procedural programming style (as in C, Java, Python)
- Problem arose from medical image processing.
- A functional approach using Haskell is in Bird (1998, page 134), Bird (2014, page 127, 133) — a variant on this called the *Not the maximum segment sum* is given in Bird (2010, Page 73) — both of these *derive* a linear time program from the (n^3) initial specification
- See Wikipedia: Maximum subarray problem
- See Rosetta Code: Greatest subsequential sum
- Here is the same program but modified to allow lists that may only have negative numbers
- The complexity $T(n)$ function will be slightly different
- but the Big O complexity will be the same

```python
def maxSubSeg01(xs) :
  n = len(xs)
  maxSoFar = xs[0]
  for i in range(1,n) :
    for j in range(i + 1, n + 1) :
      s = sum(xs[i:j])
      maxSoFar = max(maxSoFar, s)
  return maxSoFar
```

- Complexity function $T(n)$ for maxSubSeg01()
- Two initial assignments
- The outer loop will be executed $(n - 1)$ times,
- Hence the inner loop is executed

$$(n-1) + (n-2) + \ldots + 2 + 1 = \frac{(n-1)}{2} \times n$$

- Assume sum() takes n assignments

- Hence $T(n) = 2 + (n+2) \times \left(\frac{(n-1)}{2} \times n\right)$

  $= 2 + (n+2) \times \left(\frac{n^2}{2} - \frac{n}{2}\right)$

  $= 2 + \frac{1}{2}n^3 - \frac{1}{2}n^2 + n^2 - n$

  $= \frac{1}{2}n^3 + \frac{1}{2}n^2 - n + 2$

- Hence $O(n^3)$
- **Developing a better algorithm**
- Assume we know the solution (maxSoFar) for xs[0..(i - 1)]
- We extend the solution to xs[0..i] as follows:
- The maximum segment will be either maxSoFar
- or the sum of a sublist ending at i (maxToHere) if it is bigger
- This reasoning is similar to divide and conquer in binary search or Dynamic programming (see Unit 5)
- Keep track of both maxSoFar and maxToHere — the Eureka step
- **Developing a better algorithm** maxSubSeg02()

```python
def maxSubSeg02(xs) :
  maxToHere = xs[0]
  maxSoFar  = xs[0]
  for x in xs[1:] :
    # Invariant: maxToHere, maxSoFar OK for xs[0..(i-1)]
    maxToHere = max(x, maxToHere + x)
    maxSoFar  = max(maxSoFar, maxToHere)
  return maxSoFar
```

- Complexity function $T(n) = 2 + 2n$
- Hence $O(n)$
- What if we want more information ?
- Return the (or a) segment with max sum and position in list

```python
def maxSubSeg03(xs) :
  maxSoFar = maxToHere = xs[0]
  startIdx, endIdx, startMaxToHere = 0, 0, 0
  for i, x in enumerate(xs) :
    if maxToHere + x < x :
      maxToHere = x
      startMaxToHere = i
    else :
      maxToHere = maxToHere + x

    if maxSoFar < maxToHere :
      maxSoFar = maxToHere
      startIdx, endIdx = startMaxToHere, i

  return (maxSoFar,xs[startIdx:endIdx+1],startIdx,endIdx)
```

- **Developing a better algorithm** maxSubSeg03()
- Complexity function worst case $T(n) = 2 + 3 + (2 + 3)n$
- Hence still $O(n)$

- Note Python assignments, enumerate() and tuple
- Sample data and output

```
egList = [-2,1,-3,4,-1,2,1,-5,4]

egList01 = [-1,-1,-1]

egList02 = [1,2,3]

assert maxSubSeg03(egList) == (6, [4, -1, 2, 1], 3, 6)

assert maxSubSeg03(egList01) == (-1, [-1], 0, 0)

assert maxSubSeg03(egList02) == (7, [1, 2, 3], 0, 2)
```

ToC

6.2 Complexity & Python Data Types

Lists

Operation	Notation	Average	Amortized Worst
Get item	`x = xs[i]`	$O(1)$	$O(1)$
Set item	`xs[i] = x`	$O(1)$	$O(1)$
Append	`xs = ys + zs`	$O(1)$	$O(1)$
Copy	`xs = ys[:]`	$O(n)$	$O(n)$
Pop last	`xs.pop()`	$O(1)$	$O(1)$
Pop other	`xs.pop(i)`	$O(k)$	$O(k)$
Insert(i,x)	`xs[i:i] = [x]`	$O(n)$	$O(n)$
Delete item	`del xs[i:i+1]`	$O(n)$	$O(n)$
Get slice	`xs = ys[i:j]`	$O(k)$	$O(k)$
Set slice	`xs[i:j] = ys`	$O(k+n)$	$O(k+n)$
Delete slice	`xs[i:j] = []`	$O(n)$	$O(n)$
Member	`x in xs`	$O(n)$	
Get length	`n = len(xs)`	$O(1)$	$O(1)$
Count(x)	`n = xs.count(x)`	$O(n)$	$O(n)$

- Source `https://wiki.python.org/moin/TimeComplexity`
- See `https://docs.python.org/3/library/stdtypes.html#sequence-types-list-tuple-range`

Bags

```
class Bag:

  def __init__(self):
    self.list = []

  def add(self, item):
    self.list.append(item)

  def remove(self, item):
    self.list.remove(item)

  def contains(self, item):
    return item in self.list

  def count(self, item):
    return self.list.count(item)

  def size(self):
    return len(self.list)
```

```
    def __str__(self):
      return str(self.list)
```

Information Retrieval Functions

- **Term Frequency**, `tf`, takes a string, `term`, and a Bag, `document`

  returns occurrences of `term` divided by total strings in `document`

- **Inverse Document Frequency**, `idf`, takes a string, `term`, and a list of Bags, `documents`

  returns `log`(`total`/(1 + `containing`)) — `total` is total number of Bags, `containing` is the number of Bags containing `term`

- **tf-idf**, `tf_idf`, takes a string, `term`, and a list of Bags, `documents`

  returns a sequence $[r_0, r_1, \ldots, r_{n-1}]$ such that $r_i =$ `tf`(`term`, d_i)$\times$`idf`(`term`, `documents`)

ToC

6.3 Definitions and Rules for Complexity

6.3.1 Big-O and Big-Theta Definitions

- We compare the functions implementing algorithms by looking at the asymptotic behaviour of the functions for large inputs.
- If f and g are functions taking taking natural numbers as input (the problem size) and returning nonnegative results (the effort required in the calculations.)
- f is of *order* g and write $f = \Theta(g)$, if there are positive constants k_1 and k_2 and a natural number n_0 such that

  $k_1 g(n) \leqslant f(n) \leqslant k_2 g(n)$ for all $n > n_0$

  This means that some multipliers times $g(n)$ provide upper and lower bounds to $f(n)$
- If we only wanted an upper bound on the values of a function, then you can use Big-O notation.
- We say f is of *order at most* g and write $f = O(g)$, if there is a positive constant k and a natural number n_0 such that

  $f(n) \leqslant kg(n)$ for all $n > n_0$
- Note that the notation is heavily abused:

  Many authors use Big-O notation when they really mean Big-Θ notation

  We really should define the Θ notation to say that $\Theta(g)$ denotes the set of all functions f with the stated property and write $f \in \Theta(g)$ — however the use of $f = \Theta(g)$ is traditional
- The next section gives some rules for manipulating the notation to calculate overall complexities of functions from their component parts — this also abuses the notation for equality

Based on Bird and Gibbons (2020, page 25) *Algorithm Design with Haskell* and Graham et al. (1994, page 450) *Concrete Mathematics: A Foundation for Computer Science*

ToC

6.3.2 Big-O and Big-Theta Rules

- $n^p = O(n^q)$ where $p \leqslant q$

  This has some surprising consequences — $n = O(n)$ and $n = O(n^2)$ — remember Big-O just gives upper bounds.

- $O(f(n)) + O(g(n)) = O(|f(n)| + |g(n)|)$
- $\Theta(n^p) + \Theta(n^q) = \Theta(n^q)$ where $p \leqslant q$
- $f(n) = \Theta(f(n))$
- $c \cdot \Theta(f(n)) = \Theta(f(n))$ if c is constant
- $\Theta(\Theta(f(n))) = \Theta(f(n))$
- $\Theta(f(n))\Theta(g(n)) = \Theta(f(n)g(n))$
- $\Theta(f(n)g(n)) = f(n)\Theta(g(n))$

ToC

6.3.3 Big-Theta Rules — Example

```
def numVowels(txt : str) -> int ;
  """Find the number of vowels in text

  """

  vowelCount = 0
  vowels = "aeiouAEIOU"

  for ch in txt :
    if ch in vowels :
      vowelCount = vowelCount + 1
  return vowelCount
```

- The rules give

  $\Theta(1) + \Theta(1) + \Theta(n) \times \Theta(|\text{vowels}|) \times \Theta(1)$

  where $n = |\text{txt}|$

- Since $|\text{vowels}| = 10$ the overall complexity is $\Theta(n)$

ToC

ToC

6.4 List Comprehensions

List Comprehensions — Python

- **List Comprehensions** (tutorial), **List Comprehensions** (reference) provide a concise way of performing calculations over lists (or other iterables)

- Example: Square the even numbers between 0 and 9

```
Python3>>> [x ** 2 for x in range(10) if x % 2 == 0]
[0, 4, 16, 36, 64]
```

  - Example: List all pairs of integers (x, y) such that $x < 4$, $y < 4$ and x is divisible by 2 and y is divisible by 3

```
Python3>>> [(x,y) for x in range(4)
...                for y in range(4)
...                if x % 2 == 0
...                   and y % 3 == 0]
[(0, 0), (0, 3), (2, 0), (2, 3)]
Python3>>>
```

- In general

```
  [expr for target1 in iterable1 if cond1
        for target2 in iterable2 if cond2 ...
        for targetN in iterableN if condN ]
```

- Lots example usage in the algorithms below

List Comprehensions — Haskell

- **List Comprehensions** provide a concise way of performing calculations over lists
- Example: Square the even numbers between 0 and 9

```
GHCi> [x^2 | x <- [0..9], x `mod` 2 == 0]
[0,4,16,36,64]
GHCi>
```

- In general

```
  [expr | qual1, qual2,..., qualN]
```

- The qualifiers `qual` can be
  - Generators `pattern <- list`
  - Boolean guards — acting as filters
  - Local declarations with `let decls` for use in `expr` and later generators and boolean guards

Activity 1 (a) Stop Words Filter

- Stop words are the most common words that most search engines avoid: `'a','an','the','tha`
- Using list comprehensions, write a function `filterStopWords` that takes a list of words and filters out the stop words
- Here is the initial code

```
sentence \
 = "the quick brown fox jumps over the lazy dog"

words = sentence.split()

wordsTest \
 = (words == ['the', 'quick', 'brown'
             , 'fox', 'jumps', 'over'
             , 'the', 'lazy', 'dog'])

stopWords \
 = ['a','an','the','that']
```

Go to Answer

Activity 1 (a) Stop Words Filter

```
sentence \
 = "the␣quick␣brown␣fox␣jumps␣over␣the␣lazy␣dog"

words = sentence.split()

wordsTest \
 = (words == ['the', 'quick', 'brown'
             , 'fox', 'jumps', 'over'
             , 'the', 'lazy', 'dog'])

stopWords \
 = ['a','an','the','that']
```

- Notice the Python Explicit line joining with (\<nl>) and Python Implicit line joining with ((...))
- The backslash (\) must be followed by an end of line character (<nl>)
- The ('␣') symbol represents a space (see Unicode U+2423 Open Box)

Go to Answer

Activity 1 (b) Transpose Matrix

- A matrix can be represented as a list of rows of numbers
- We *transpose* a matrix by swapping columns and rows
- Here is an example

```
matrixA \
 = [[1, 2, 3, 4]
   ,[5, 6, 7 ,8]
   ,[9, 10, 11, 12]]

matATr \
 = [[1, 5, 9]
   ,[2, 6, 10]
   ,[3, 7, 11]
   ,[4, 8, 12]]
```

- Using list comprehensions, write a function `transMat`, to transpose a matrix

Go to Answer

Activity 1 (c) List Pairs in Fair Order

- Write a function which takes a pair of positive integers and outputs a list of all possible pairs in those ranges
- If we do this in the simplest way we get a bias to one argument
- Here is an example of a bias to the second argument

```
yBiasLstTest \
 = (yBiasListing(5,5)
     == [(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)
        , (1, 0), (1, 1), (1, 2), (1, 3), (1, 4)
        , (2, 0), (2, 1), (2, 2), (2, 3), (2, 4)
        , (3, 0), (3, 1), (3, 2), (3, 3), (3, 4)
        , (4, 0), (4, 1), (4, 2), (4, 3), (4, 4)])
```

Go to Answer

Activity 1 (c) List Pairs in Fair Order

- Rewrite the function which takes a pair of positive integers and outputs a list of all possible pairs in those ranges
- The output should treat each argument *fairly* — any initial prefix should have roughly the same number of instances of each argument
- Here is an example output

```
fairLstTest \
 = (fairListing(5,5)
     == [(0, 0)
        , (0, 1), (1, 0)
        , (0, 2), (1, 1), (2, 0)
        , (0, 3), (1, 2), (2, 1), (3, 0)
        , (0, 4), (1, 3), (2, 2), (3, 1), (4, 0)])
```

Go to Answer

Activity 1 (c) List Pairs in Fair Order

- Rewrite the function which takes a pair of positive integers and outputs a list of lists of all possible pairs in those ranges
- The output should treat each argument *fairly* — any initial prefix should have roughly the same number of instances of each argument — further, the output should be segment by each initial prefix (see example below)
- Here is an example output

```
fairLstATest \
 = (fairListingA(5,5)
     == [[(0, 0)]
        , [(0, 1), (1, 0)]
        , [(0, 2), (1, 1), (2, 0)]
        , [(0, 3), (1, 2), (2, 1), (3, 0)]
        , [(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)]])
```

Go to Answer

6.4.1 Complexity of List Comprehensions

- Note that list comprehensions are not in M269
- See Complexity of a List Comprehension

```
[f(e) for e in row for row in mat]
```

- Suppose $f = \Theta(g)$ with n elements in a row and m rows
- Then complexity is $\Theta(g(e)) \times \Theta(n) \times \Theta(m) = \Theta(m \times n \times g(e))$

```
[[e**2 for e in row] for row in mat]
```

- $\Theta(e * *2) = \Theta(1)$
- Suppose n is maximum length of a row and m rows
- Then complexity is$\Theta(1) \times \Theta(n) \times \Theta(m) = \Theta(n \times m)$

ToC

Answer 1 (a) Stop Words Filter

- Answer 1 (a) Stop Words Filter

- *Write here:*

Answer 1 (a) Stop Words Filter

- Answer 1 (a) Stop Words Filter

```
def filterStopWords(words) :
  nonStopWords \
   = [word for word in words
            if word not in stopWords]
  return nonStopWords


filterStopWordsTest \
 = filterStopWords(words) \
    == ['quick', 'brown', 'fox'
      , 'jumps', 'over', 'lazy', 'dog']
```

Go to Activity

Answer 1 (b) Transpose Matrix

- Answer 1 (b) Transpose Matrix
- *Write here:*

Answer 1 (b) Transpose Matrix

- Answer 1 (b) Transpose Matrix

```
def transMat(mat) :
  rowLen = len(mat[0])
  matTr \
   = [[row[i] for row in mat] for i in range(rowLen)]
  return matTr

transMatTestA \
 = (transMat(matrixA)
    == matATr)
```

- Note that a list comprehension is a valid expression as a target expression in a list comprehension
- The code assumes every row is of the same length

Go to Activity

Answer 1 (b) Transpose Matrix

- Note the differences in the list comprehensions below

```
matrixA \
 = [[1, 2, 3, 4]
   ,[5, 6, 7 ,8]
   ,[9, 10, 11, 12]]
```

```
Python3>>> [[row[i] for row in matrixA]
...                   for i in range(4)]
[[1, 5, 9], [2, 6, 10], [3, 7, 11], [4, 8, 12]]
Python3>>> [row[i] for row in matrixA
...                  for i in range(4)]
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
Python3>>> [row[i] for i in range(4)
...                  for row in matrixA]
[1, 5, 9, 2, 6, 10, 3, 7, 11, 4, 8, 12]
Python3>>> [[row[i] for i in range(4)]
...                   for row in matrixA]
[[1, 2, 3, 4], [5, 6, 7, 8], [9, 10, 11, 12]]
```

Go to Activity

Answer 1 (b) Transpose Matrix

- Answer 1 (b) Transpose Matrix
- The Python NumPy package provides functions for N-dimensional array objects
- For transpose see numpy.ndarray.transpose

```
Python3>>> import numpy as np
Python3>>> ar = np.array([[1,2],[3,4]])
Python3>>> ar
array([[1, 2],
       [3, 4]])
Python3>>> arT = ar.transpose()
Python3>>> arT
array([[1, 3],
       [2, 4]])
Python3>>> ar
array([[1, 2],
       [3, 4]])
Python3>>> ar.shape
(2, 2)
```

Go to Activity

Answer 1 (c) List Pairs in Fair Order

- Answer 1 (c) List Pairs in Fair Order — first version
- Write here

```
yBiasLstTest \
 = (yBiasListing(5,5)
     == [(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)
        , (1, 0), (1, 1), (1, 2), (1, 3), (1, 4)
        , (2, 0), (2, 1), (2, 2), (2, 3), (2, 4)
        , (3, 0), (3, 1), (3, 2), (3, 3), (3, 4)
        , (4, 0), (4, 1), (4, 2), (4, 3), (4, 4)])
```

Go to Activity

Answer 1 (c) List Pairs in Fair Order

- Answer 1 (c) List Pairs in Fair Order
- This is the *obvious* but biased version

```
def yBiasListing(xRng,yRng) :
  yBiasLst \
   = [(x,y) for x in range(xRng)
            for y in range(yRng)]
  return yBiasLst

yBiasLstTest \
 = (yBiasListing(5,5)
     == [(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)
        , (1, 0), (1, 1), (1, 2), (1, 3), (1, 4)
        , (2, 0), (2, 1), (2, 2), (2, 3), (2, 4)
        , (3, 0), (3, 1), (3, 2), (3, 3), (3, 4)
        , (4, 0), (4, 1), (4, 2), (4, 3), (4, 4)])
```

Go to Activity

Answer 1 (c) List Pairs in Fair Order

- Answer 1 (c) List Pairs in Fair Order — second version
- Write here

```
fairLstTest \
 = (fairListing(5,5)
     == [(0, 0)
        , (0, 1), (1, 0)
        , (0, 2), (1, 1), (2, 0)
        , (0, 3), (1, 2), (2, 1), (3, 0)
        , (0, 4), (1, 3), (2, 2), (3, 1), (4, 0)])
```

Go to Activity

Answer 1 (c) List Pairs in Fair Order

- Answer 1 (c) List Pairs in Fair Order — second version
- This works by making the sum of the coordinates the same for each prefix

```
def fairListing(xRng,yRng) :
  fairLst \
   = [(x,d-x) for d in range(yRng)
              for x in range(d+1)]
  return fairLst

fairLstTest \
 = (fairListing(5,5)
     == [(0, 0)
        , (0, 1), (1, 0)
        , (0, 2), (1, 1), (2, 0)
        , (0, 3), (1, 2), (2, 1), (3, 0)
        , (0, 4), (1, 3), (2, 2), (3, 1), (4, 0)]
```

Go to Activity

Answer 1 (c) List Pairs in Fair Order

- Answer 1 (c) List Pairs in Fair Order — third version
- Write here

```
fairLstATest \
 = (fairListingA(5,5)
     == [[(0, 0)]
        , [(0, 1), (1, 0)]
        , [(0, 2), (1, 1), (2, 0)]
        , [(0, 3), (1, 2), (2, 1), (3, 0)]
        , [(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)]])
```

Go to Activity

Answer 1 (c) List Pairs in Fair Order

- Answer 1 (c) List Pairs in Fair Order — third version
- The *inner loop* is placed into its own list comprehension

```
def fairListingA(xRng,yRng) :
  fairLstA \
   = [[(x,d-x) for x in range(d+1)]
               for d in range(yRng)]
  return fairLstA

fairLstATest \
 = (fairListingA(5,5)
     == [[(0, 0)]
        , [(0, 1), (1, 0)]
        , [(0, 2), (1, 1), (2, 0)]
        , [(0, 3), (1, 2), (2, 1), (3, 0)]
        , [(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)]])
```

Go to Activity

ToC

6.5 Master Theorem for Divide-and-Conquer Recurrences

- **The Divide-and-Conquer Method**

  Many useful algorithms are recursive in structure and often follow a *divide-and-conquer method*

  They break the problem into several subproblems similar to the original problem

- The time analysis is represented by a *recurrence system*
- **References**
- Big O notation
- Master theorem
- Cormen et al. (2022, chp 4) *Algorithms*
- These notes are partly based on *M261 Mathematics in Computing* and *M263 Building Blocks of Software* and are *not* part of *M269 Algorithms, Data Structures and Computability*
- **Recurrence System**

$$\begin{aligned} T(1) &= b & (1)\\ T(n) &= bn^{\beta} + cT\left(\frac{n}{d}\right) \qquad \{n = d^{\alpha} > 1\} & (2) \end{aligned}$$

- **Typical Expansion**

n	**T(n)**
d^0	b
d^1	$bn^{\beta} + cb$
d^2	$bn^{\beta} + cb\left(\frac{n}{d}\right)^{\beta} + c^2 b$

- **General Expansion**

$$\begin{aligned} T(n) &= bn^{\beta} + cT\left(\frac{n}{d}\right)\\ &= bn^{\beta} + cb\left(\frac{n}{d}\right)^{\beta} + c^2 T\left(\frac{n}{d^2}\right)\\ &= bn^{\beta}\left(1 + \frac{c}{d^{\beta}} + \left(\frac{c}{d^{\beta}}\right)^2 + \cdots + \left(\frac{c}{d^{\beta}}\right)^{\alpha}\right)\\ T(n) &= bn^{\beta}\sum_{i=0}^{\log_d n}\left(\frac{c}{d^{\beta}}\right)^i & (3) \end{aligned}$$

- **Proof of Closed Form Equation (3)**
- For $n = 1$ equation (3) gives

$T(1) = b1^{\beta} \sum_{i=0}^{0} \left(\frac{c}{d^{\beta}}\right)^{i} = b$ which is correct (same as (1))

- Assume equation (3) holds for $n = d^{\alpha}$. Then for $n = d^{\alpha+1}$

$$\begin{aligned}
T\left(d^{\alpha+1}\right) &= cT\left(d^{\alpha}\right) + bn^{\beta} && \text{by equation (2)} \\
&= cbd^{\alpha\beta} \sum_{i=0}^{\alpha} \left(\tfrac{c}{d^{\beta}}\right)^{i} + bd^{(\alpha+1)\beta} && \text{by assumption} \\
&= \left(\tfrac{c}{d^{\beta}}\right) bd^{(\alpha+1)\beta} \sum_{i=0}^{\alpha} \left(\tfrac{c}{d^{\beta}}\right)^{i} + bd^{(\alpha+1)\beta} \\
&= bd^{(\alpha+1)\beta} \left(\sum_{i=1}^{\alpha+1} \left(\tfrac{c}{d^{\beta}}\right)^{i} + 1\right) && \text{by rearrangement} \\
&= bd^{(\alpha+1)\beta} \sum_{i=0}^{\alpha+1} \left(\tfrac{c}{d^{\beta}}\right)^{i} && \text{by rearrangement}
\end{aligned}$$

- Hence equation (3) holds for all $n = d^{\alpha}$ where $\alpha \in \mathbb{N}$

1. If $c < d^{\beta}$ then the sum converges and T(n) is $\Theta(n^{\beta})$
2. If $c = d^{\beta}$ then each term in the sum is 1 and
   T(n) is $\Theta\left(n^{\beta} \log_d n\right)$
3. If $c > d^{\beta}$ then use $\sum_{i=0}^{p} x^{i} = \dfrac{x^{p+1} - 1}{x - 1}$

$$\begin{aligned}
T(n) &= bn^{\beta} \left[\frac{\left(\frac{c}{d^{\beta}}\right)^{\log_d n+1} - 1}{\left(\frac{c}{d^{\beta}}\right) - 1} \right] \\
&= \Theta\left(n^{\beta} \left(\tfrac{c}{d^{\beta}}\right)^{\log_d n}\right) \\
&= \Theta\left(c^{\log_d n}\right) \\
&= \Theta\left(n^{\log_d c}\right) \text{ since } a^{\log_b x} = x^{\log_b a}
\end{aligned}$$

6.5.1 Master Theorem Example Usage

- **Algorithm**
- Find mid point and check
  if not equal to target, recurse on half the data
- **Timing equations**

  $T(1) \leqslant 1$

  $T(n) = T\left(\frac{n}{2}\right) + 1$

- Hence $c = 1$, $d = 2$, $\beta = 0 \rightarrow$ case (2)

  $T(n) = \Theta(\log_2 n)$

- **Algorithm**

- Best case: splitting on median of data
- Recursively sort each half
- **Timing equations**

  $T(1) \leqslant k$

  $T(n) = 2T\left(\frac{n}{2}\right) + kn$
- Hence $c = 2$, $d = 2$, $\beta = 1 \rightarrow$ case (2)

  $$T(n) = \Theta(n \log_2 n)$$
- See Averages/Median
- **Matrix Multiplication**
- Let A, B be two square matrices over a ring, $\mathcal{R}$
- Informally, a *ring* is a set with two binary operations which look similar to addition and multiplication of integers
- The problem is to implement matrix multiplication to find the matrix product $C = AB$
- Without loss of generality, we may assume that A, and B have sizes which are powers of 2 — if A, and B were not of this size, they could be padded with rows or columns of zeroes
- The Strassen algorithm partitions A, B and C into equally sized blocks

  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ $\quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ $\quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$

  with A_{ij}, B_{ij}, $C_{ij} \in \mathrm{Mat}_{2^{n-1} \times 2^{n-1}}(\mathcal{R})$
- The usual (naive, standard) algorithm gives

  $$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$
  $$= \begin{pmatrix} A_{11} \times B_{11} + A_{12} \times B_{21} & A_{11} \times B_{12} + A_{12} \times B_{22} \\ A_{21} \times B_{11} + A_{22} \times B_{21} & A_{21} \times B_{12} + A_{22} \times B_{22} \end{pmatrix}$$
- This as 8 multiplications and if we assume multiplication is more expensive than addition then the time complexity is $\Theta(n^3)$
- The Strassen algorithm rearranges the calculation

  $M_1 = (A_{11} + A_{22}) \times (B_{11} + B_{22})$

  $M_2 = (A_{21} + A22) \times B_{11}$

  $M_3 = A_{11} \times (B_{12} - B_{22})$

  $M_4 = A_{22} \times (B_{21} - B_{11})$

  $M_5 = (A_{11} + A12) \times B_{22}$

  $M_6 = (A_{21} - A_{11}) \times (B_{11} + B_{12})$

  $M_7 = (A_{12} - A_{22}) \times (B_{21} + B_{22})$

- We now express the C_{ij} in terms of the M_k

  $\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$
  $= \begin{pmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 - M_2 + M_3 + M_6 \end{pmatrix}$

- **Strassen Matrix Multiplication Timing Equations**

  $T(n) = 7T\left(\frac{n}{2}\right) + \frac{18}{4}n^2$

  $T(1) \leqslant \frac{18}{4}$

- This is derived from the 7 multiplications and 18 additions or subtractions
- $c = 7$, $d = 2$, $\beta = 2 \rightarrow$ case (3)

  $$T(n) = \Theta\left(n^{\log_2 7}\right) = \Theta\left(n^{2.8}\right)$$

ToC

7 Exponentials and Logarithms

7.1 Exponentials and Logarithms — Definitions

- **Exponential function** $y = a^x$ or $f(x) = a^x$
- $a^n = a \times a \times \cdots \times a$ (n a terms)
- **Logarithm** reverses the operation of exponentiation
- $\log_a y = x$ means $a^x = y$
- $\log_a 1 = 0$
- $\log_a a = 1$
- Method of logarithms propounded by John Napier from 1614
- Log Tables from 1617 by Henry Briggs
- Slide Rule from about 1620–1630 by William Oughtred of Cambridge
- *Logarithm* from Greek *logos* ratio, and *arithmos* number Chambers (2014) *Chambers Dictionary*

ToC

7.2 Rules of Indices

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$
3. $a^{-m} = \dfrac{1}{a^m}$
4. $a^{\frac{1}{m}} = \sqrt[m]{a}$
5. $(a^m)^n = a^{mn}$

6. $a^{\frac{n}{m}} = \sqrt[m]{a^n}$
7. $a^0 = 1$ where $a \neq 0$

- **Exercise** Justify the above rules
- What should 0^0 evaluate to ?
- See Wikipedia: Exponentiation
- The *justification* above probably only worked for whole or *rational* numbers — see later for exponents with *real* numbers (and the value of *logarithms*, *calculus*...)

ToC

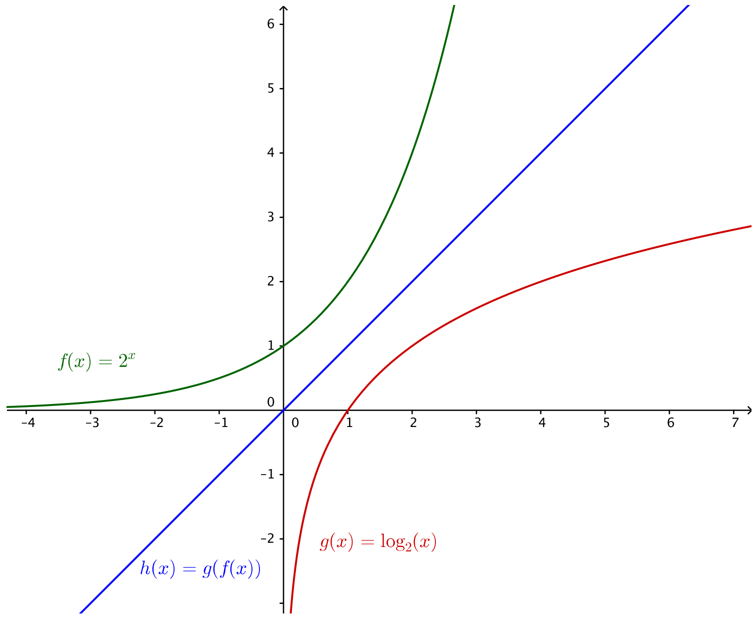
7.3 Logarithms — Motivation

- Make arithmetic easier — turns multiplication and division into addition and subtraction (see later)
- Complete the range of elementary functions for differentiation and integration
- An elementary function is a function of one variable which is the composition of a finite number of arithmetic operations ((+), (−), (×), (÷)), exponentials, logarithms, constants, and solutions of algebraic equations (a generalization of nth roots).
- The elementary functions include the trigonometric and hyperbolic functions and their inverses, as they are expressible with complex exponentials and logarithms.
- See A Level FP2 for Euler's relation $\mathrm{e}^{i\theta} = \cos\theta + \mathrm{i}\sin\theta$
- In A Level C3, C4 we get $\int \frac{1}{x} = \log_{\mathrm{e}} |x| + C$
- e is Euler's number 2.71828...

ToC

7.4 Exponentials and Logarithms — Graphs

- See GeoGebra file `expLog.ggb`



ToC

7.5 Laws of Logarithms

- **Multiplication law** $\log_a xy = \log_a x + \log_a y$
- **Division law** $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
- **Power law** $\log_a x^k = k \log_a x$
- **Proof of Multiplication Law**

$$\begin{aligned}
x &= a^{\log_a x} \\
y &= a^{\log_a y} && \text{by definition of log} \\
xy &= a^{\log_a x} \times a^{\log_a y} \\
&= a^{\log_a x + \log_a y} && \text{by laws of indices}
\end{aligned}$$

Hence $\log_a xy = \log_a x + \log_a y$ by definition of log

ToC

7.6 Arithmetic and Inverses

- *Notation helps or maybe not ?*
- **Addition** $\text{add}(b, x) = x + b$
- **Subtraction** $\text{sub}(b, x) = x - b$
- Inverse $\text{sub}(b, \text{add}(b, x)) = (x + b) - b = x$
- **Multiplication** $\text{mul}(b, x) = x \times b$
- **Division** $\text{div}(b, x) = x \div b = \frac{x}{b} = x/b$

- Inverse $\mathrm{div}(b, \mathrm{mul}(b, x)) = (x \times b) \div b = \frac{(x \times b)}{b} = x$
- **Exponentiation** $\exp(b, x) = b^x$
- **Logarithm** $\log(b, x) = \log_b x$
- Inverse $\log(b, \exp(b, x)) = \log_b(b^x) = x$
- *What properties do the operations have that work (or not) with the notation ?*

Arithmetic Operations — Commutativity and Associativity

- **Commutativity** $x \circledast y = y \circledast x$
- **Associativity** $(x \circledast y) \circledast z = x \circledast (y \circledast z)$
- $(+)$ and $(\times)$ are *semantically* commutative and associative — so we can leave the brackets out
- $(-)$ and $(\div)$ are not
- Evaluate $(3 - (2 - 1))$ and $((3 - 2) - 1)$
- Evaluate $(3/(2/2))$ and $((3/2)/2)$
- We have the *syntactic* ideas of left (and right) associativity
- We choose $(-)$ and $(\div)$ to be left associative
- $3 - 2 - 1$ means $((3 - 2) - 1)$
- $3/2/2$ means $((3/2)/2)$
- **Operator precedence** is also a choice (remember BIDMAS or BODMAS ?)
- If in doubt, put the brackets in

Exponentials and Logarithm — Associativity

- What should 2^{3^4} mean ?
- Let $b \wedge x \equiv b^x$
- Evaluate $(2 \wedge 3) \wedge 4$ and $2 \wedge (3 \wedge 4)$
- Evaluate $c = \log_b(\log_b((b \wedge b) \wedge x))$
- Evaluate $d = \log_b(\log_b(b \wedge (b \wedge x)))$
- Beware spreadsheets Excel and LibreOffice here
- $(2^3)^4 = 2^{12}$ and $2^{3^4} = 2^{81}$
- Exponentiation is not semantically associative
- We *choose* the syntactic left or right associativity to make the syntax nicer.
- Evaluate $c = \log_b(\log_b((b \wedge b) \wedge x))$
- $c = \log_b(x \log_b(b^b)) = \log_b(x \cdot (b \log_b b)) = \log_b(x \cdot b \cdot 1)$
- Hence $c = \log_b x + \log_b b = \log_b x + 1$
- Not symmetrical (unless b and x are both 2)

- Evaluate $d = \log_b(\log_b(b \wedge (b \wedge x)))$
- $d = \log_b((b \wedge x)(\log_b b)) = \log_b((b \wedge x) \times 1)$
- Hence $d = \log_b(b \wedge x) = x(\log_b b) = x$
- Which is what we want — so exponentiation is *chosen* to be right associative

ToC

7.7 Change of Base

- **Change of base**

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Proof: Let $y = \log_a x$

$$\begin{aligned} a^y &= x \\ \log_b a^y &= \log_b x \\ y \log_b a &= \log_b x \\ y &= \frac{\log_b x}{\log_b a} \end{aligned}$$

- Given x, $\log_b x$, find the base b
  - $b = x^{\frac{1}{\log_b x}}$
- $\log_a b = \dfrac{1}{\log_b a}$

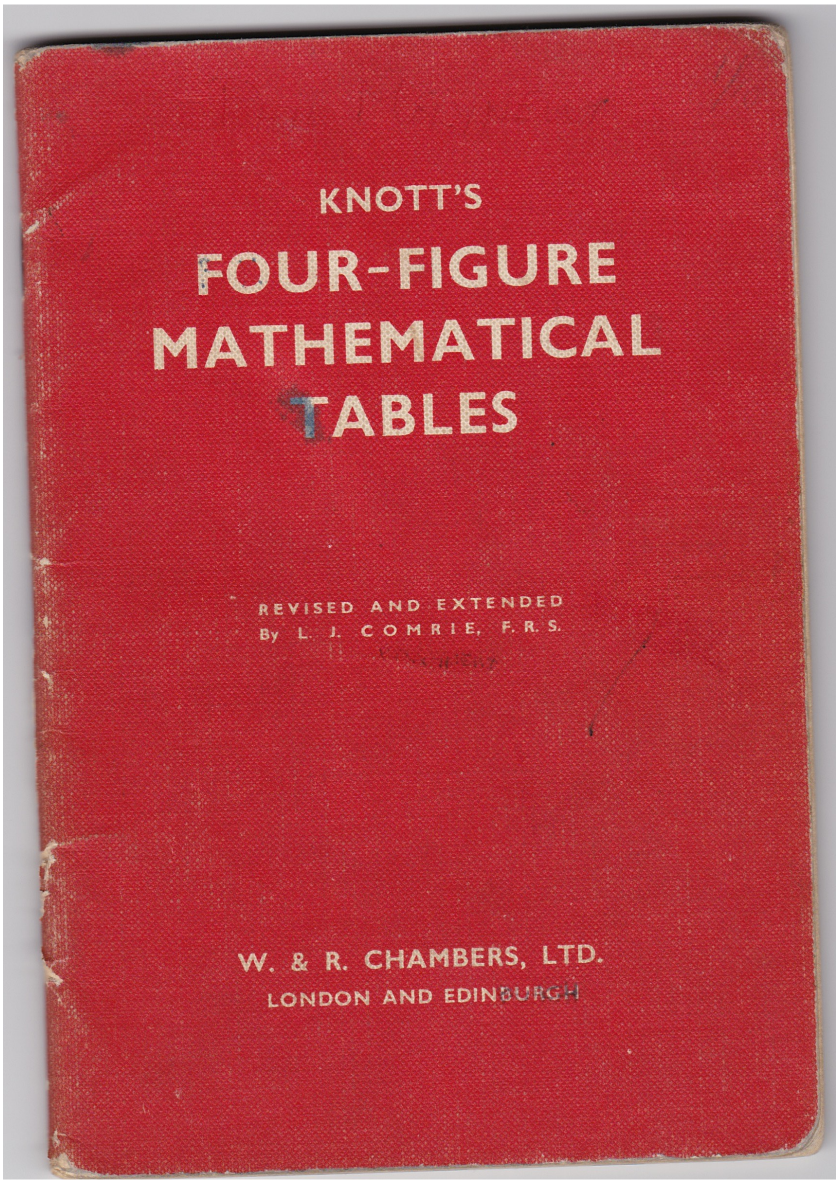
ToC

8 Before Calculators and Computers

- We had computers before 1950 — they were *humans* with pencil, paper and some further aids:
- **Slide rule** invented by William Oughtred in the 1620s — major calculating tool until pocket calculators in 1970s
- **Log tables** in use from early 1600s — method of logarithms propounded by John Napier
- **Logarithm** from Greek *logos* ratio, and *arithmos* number

8.1 Log Tables

Knott's Four-Figure Mathematical Tables



Logarithms of Numbers

x	0	1	2	3	4	5	6	7	8	9	Δ	1	2	3	4	5	6	7	8	9
10	·0000	0043	0086	0128	0170	0212					42	4	8	13	17	21	25	29	34	38
						0212	0253	0294	0334	0374	40	4	8	12	16	20	24	28	32	36
11	·0414	0453	0492	0531	0569	0607					39	4	8	12	16	19	23	27	31	35
						0607	0645	0682	0719	0755	37	4	7	11	15	18	22	26	30	33
12	·0792	0828	0864	0899	0934	0969					35	4	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	34	3	7	10	14	17	20	24	27	31
13	·1139	1173	1206	1239	1271	1303					33	3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	32	3	6	10	13	16	19	22	26	29
14	·1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30	3	6	9	12	15	18	21	24	27
15	·1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28	3	6	8	11	14	17	20	22	25
16	·2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26	3	5	8	10	13	16	18	21	23
17	·2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25	2	5	7	10	12	15	17	20	22
18	·2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	23	2	5	7	9	12	14	16	19	21
19	·2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22	2	4	7	9	11	13	15	18	20
20	·3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21	2	4	6	8	11	13	15	17	19
21	·3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20	2	4	6	8	10	12	14	16	18
22	·3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19	2	4	6	8	10	11	13	15	17
23	·3617	3636	3655	3674	3692	3711	3729	3747	3766	3784		2	4	6	7	9	11	13	15	17
24	·3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18	2	4	5	7	9	11	13	14	16
25	·3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17	2	3	5	7	9	10	12	14	15
26	·4150	4166	4183	4200	4216	4232	4249	4265	4281	4298		2	3	5	6	8	10	11	13	14
27	·4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16	2	3	5	6	8	9	11	13	14
28	·4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15	2	3	5	6	8	9	11	12	14
29	·4624	4639	4654	4669	4683	4698	4713	4728	4742	4757		1	3	4	6	7	9	10	12	13
30	·4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14	1	3	4	6	7	8	10	11	13
31	·4914	4928	4942	4955	4969	4983	4997	5011	5024	5038		1	3	4	5	7	8	10	11	12
32	·5051	5065	5079	5092	5105	5119	5132	5145	5159	5172		1	3	4	5	7	8	9	11	12
33	·5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13	1	3	4	5	7	8	9	10	12
34	·5315	5328	5340	5353	5366	5378	5391	5403	5416	5428		1	3	4	5	6	8	9	10	11
35	·5441	5453	5465	5478	5490	5502	5514	5527	5539	5551		1	2	4	5	6	7	8	10	11
36	·5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12	1	2	4	5	6	7	8	10	11
37	·5682	5694	5705	5717	5729	5740	5752	5763	5775	5786		1	2	3	5	6	7	8	9	10
38	·5798	5809	5821	5832	5843	5855	5866	5877	5888	5899		1	2	3	5	6	7	8	9	10
39	·5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11	1	2	3	4	6	7	8	9	10
40	·6021	6031	6042	6053	6064	6075	6085	6096	6107	6117		1	2	3	4	5	6	7	9	10
41	·6128	6138	6149	6160	6170	6180	6191	6201	6212	6222		1	2	3	4	5	6	7	8	9
42	·6232	6243	6253	6263	6274	6284	6294	6304	6314	6325		1	2	3	4	5	6	7	8	9
43	·6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10	1	2	3	4	5	6	7	8	9
44	·6435	6444	6454	6464	6474	6484	6493	6503	6513	6522		1	2	3	4	5	6	7	8	9
45	·6532	6542	6551	6561	6571	6580	6590	6599	6609	6618		1	2	3	4	5	6	7	8	9
46	·6628	6637	6646	6656	6665	6675	6684	6693	6702	6712		1	2	3	4	5	6	7	7	8
47	·6721	6730	6739	6749	6758	6767	6776	6785	6794	6803		1	2	3	4	5	5	6	7	8
48	·6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9	1	2	3	4	5	5	6	7	8
49	·6902	6911	6920	6928	6937	6946	6955	6964	6972	6981		1	2	3	4	4	5	6	7	8

USEFUL CONSTANTS WITH THEIR LOGARITHMS

	No.	Log.		No.	Log.		No.	Log.
π	3·14159	0·4971	1 radian	57°·296	1·7581	e	2·71828	0·4343
$\frac{1}{\pi}$	0·3183	$\bar{1}$·5029		3437′·7	3·5363	M	0·4343	$\bar{1}$·6378
π^2	9·8696	0·9943		206265″	5·3144	$\frac{1}{M}$	2·3026	0·3622
$\sqrt{\pi}$	1·7725	0·2486	arc 1°	0·017 453 293	$\bar{2}$·2419			
$\frac{4}{3}\pi$	4·1888	0·6221	arc 1′	0·000 290 888	$\bar{4}$·4637			
			arc 1″	0·000 004 848	$\bar{6}$·6856			

$\log_e x = \frac{1}{M} \cdot \log_{10} x$

$\log_{10} x = M \cdot \log_e x$

x	0	1	2	3	4	5	6	7	8	9	Δ	1	2	3	4	5	6	7	8	9
50	·6990	6998	7007	7016	7024	7033	7042	7050	7059	7067		1	2	3	3	4	5	6	7	8
51	·7076	7084	7093	7101	7110	7118	7126	7135	7143	7152		1	2	3	3	4	5	6	7	8
52	·7160	7168	7177	7185	7193	7202	7210	7218	7226	7235		1	2	2	3	4	5	6	7	7
53	·7243	7251	7259	7267	7275	7284	7292	7300	7308	7316		1	2	2	3	4	5	6	6	7
54	·7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8	1	2	2	3	4	5	6	6	7
55	·7404	7412	7419	7427	7435	7443	7451	7459	7466	7474		1	2	2	3	4	5	5	6	7
56	·7482	7490	7497	7505	7513	7520	7528	7536	7543	7551		1	2	2	3	4	5	5	6	7
57	·7559	7566	7574	7582	7589	7597	7604	7612	7619	7627		1	2	2	3	4	5	5	6	7
58	·7634	7642	7649	7657	7664	7672	7679	7686	7694	7701		1	2	2	3	4	4	5	6	7
59	·7709	7716	7723	7731	7738	7745	7752	7760	7767	7774		1	1	2	3	4	4	5	6	7
60	·7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7	1	1	2	3	4	4	5	6	6
61	·7853	7860	7868	7875	7882	7889	7896	7903	7910	7917		1	1	2	3	4	4	5	6	6
62	·7924	7931	7938	7945	7952	7959	7966	7973	7980	7987		1	1	2	3	3	4	5	6	6
63	·7993	8000	8007	8014	8021	8028	8035	8041	8048	8055		1	1	2	3	3	4	5	5	6
64	·8062	8069	8075	8082	8089	8096	8102	8109	8116	8122		1	1	2	3	3	4	5	5	6
65	·8129	8136	8142	8149	8156	8162	8169	8176	8182	8189		1	1	2	3	3	4	5	5	6
66	·8195	8202	8209	8215	8222	8228	8235	8241	8248	8254		1	1	2	3	3	4	5	5	6
67	·8261	8267	8274	8280	8287	8293	8299	8306	8312	8319		1	1	2	3	3	4	4	5	6
68	·8325	8331	8338	8344	8351	8357	8363	8370	8376	8382		1	1	2	3	3	4	4	5	6
69	·8388	8395	8401	8407	8414	8420	8426	8432	8439	8445		1	1	2	3	3	4	4	5	6
70	·8451	8457	8463	8470	8476	8482	8488	8494	8500	8506		1	1	2	2	3	4	4	5	6
71	·8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6	1	1	2	2	3	4	4	5	5
72	·8573	8579	8585	8591	8597	8603	8609	8615	8621	8627		1	1	2	2	3	4	4	5	5
73	·8633	8639	8645	8651	8657	8663	8669	8675	8681	8686		1	1	2	2	3	4	4	5	5
74	·8692	8698	8704	8710	8716	8722	8727	8733	8739	8745		1	1	2	2	3	4	4	5	5
75	·8751	8756	8762	8768	8774	8779	8785	8791	8797	8802		1	1	2	2	3	3	4	5	5
76	·8808	8814	8820	8825	8831	8837	8842	8848	8854	8859		1	1	2	2	3	3	4	5	5
77	·8865	8871	8876	8882	8887	8893	8899	8904	8910	8915		1	1	2	2	3	3	4	4	5
78	·8921	8927	8932	8938	8943	8949	8954	8960	8965	8971		1	1	2	2	3	3	4	4	5
79	·8976	8982	8987	8993	8998	9004	9009	9015	9020	9025		1	1	2	2	3	3	4	4	5
80	·9031	9036	9042	9047	9053	9058	9063	9069	9074	9079		1	1	2	2	3	3	4	4	5
81	·9085	9090	9096	9101	9106	9112	9117	9122	9128	9133		1	1	2	2	3	3	4	4	5
82	·9138	9143	9149	9154	9159	9165	9170	9175	9180	9186		1	1	2	2	3	3	4	4	5
83	·9191	9196	9201	9206	9212	9217	9222	9227	9232	9238		1	1	2	2	3	3	4	4	5
84	·9243	9248	9253	9258	9263	9269	9274	9279	9284	9289		1	1	2	2	3	3	4	4	5
85	·9294	9299	9304	9309	9315	9320	9325	9330	9335	9340		1	1	2	2	3	3	4	4	5
86	·9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5	1	1	2	2	3	3	4	4	5
87	·9395	9400	9405	9410	9415	9420	9425	9430	9435	9440		0	1	1	2	2	3	3	4	4
88	·9445	9450	9455	9460	9465	9469	9474	9479	9484	9489		0	1	1	2	2	3	3	4	4
89	·9494	9499	9504	9509	9513	9518	9523	9528	9533	9538		0	1	1	2	2	3	3	4	4
90	·9542	9547	9552	9557	9562	9566	9571	9576	9581	9586		0	1	1	2	2	3	3	4	4
91	·9590	9595	9600	9605	9609	9614	9619	9624	9628	9633		0	1	1	2	2	3	3	4	4
92	·9638	9643	9647	9652	9657	9661	9666	9671	9675	9680		0	1	1	2	2	3	3	4	4
93	·9685	9689	9694	9699	9703	9708	9713	9717	9722	9727		0	1	1	2	2	3	3	4	4
94	·9731	9736	9741	9745	9750	9754	9759	9763	9768	9773		0	1	1	2	2	3	3	4	4
95	·9777	9782	9786	9791	9795	9800	9805	9809	9814	9818		0	1	1	2	2	3	3	4	4
96	·9823	9827	9832	9836	9841	9845	9850	9854	9859	9863		0	1	1	2	2	3	3	4	4
97	·9868	9872	9877	9881	9886	9890	9894	9899	9903	9908		0	1	1	2	2	3	3	4	4
98	·9912	9917	9921	9926	9930	9934	9939	9943	9948	9952		0	1	1	2	2	3	3	4	4
99	·9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4	0	1	1	2	2	2	3	3	4

Only the decimal portion (*mantissa*) of each logarithm is shown in this table. The integral portion (*characteristic*) must be determined independently.

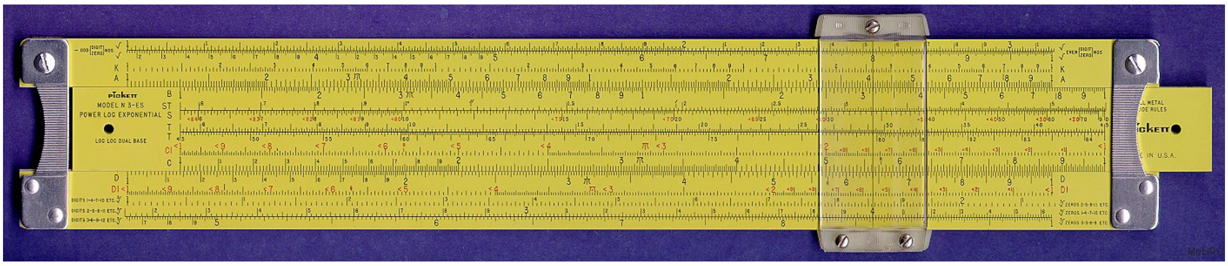
Antilogarithms

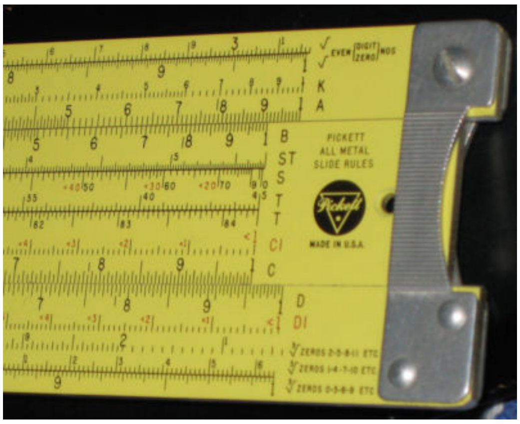
x	0	1	2	3	4	5	6	7	8	9	Δ	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	2	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045		0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069		0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094		0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119		0	1	1	1	1	1	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146		0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172		0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199		0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227		0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256		0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285		0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	3	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346		0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377		0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409		0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442		0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476		0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510		0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545		0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581		0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618		0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656		0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694		0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	4	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774		0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816		0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858		0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901		0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945		0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991		0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037		0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084		0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133		0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	5	1	1	2	2	3	3	4	4	5
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234		1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286		1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339		1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393		1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449		1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506		1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564		1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	6	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685		1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748		1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812		1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877		1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944		1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013		1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	7	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155		1	1	2	3	4	4	5	6	6

x	0	1	2	3	4	5	6	7	8	9	Δ	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228		1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304		1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381		1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459		1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	8	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622		1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707		1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793		1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882		1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	9	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064		1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159		1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256		1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	10	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457		1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560		1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667		1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	11	1	2	3	4	6	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887		1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000		1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117		1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	12	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358		1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483		1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610		1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	13	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875		1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012		1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	14	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295		1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442		1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	15	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745		2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902		2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	16	2	3	5	6	8	10	11	13	14
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228		2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	17	2	3	5	7	9	10	12	14	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568		2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745		2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	18	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110		2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	19	2	4	6	8	10	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492		2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690		2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	20	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099		2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	21	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528		2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	22	2	4	7	9	11	13	15	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	23	2	5	7	9	11	14	16	18	21

8.2 Slide Rules

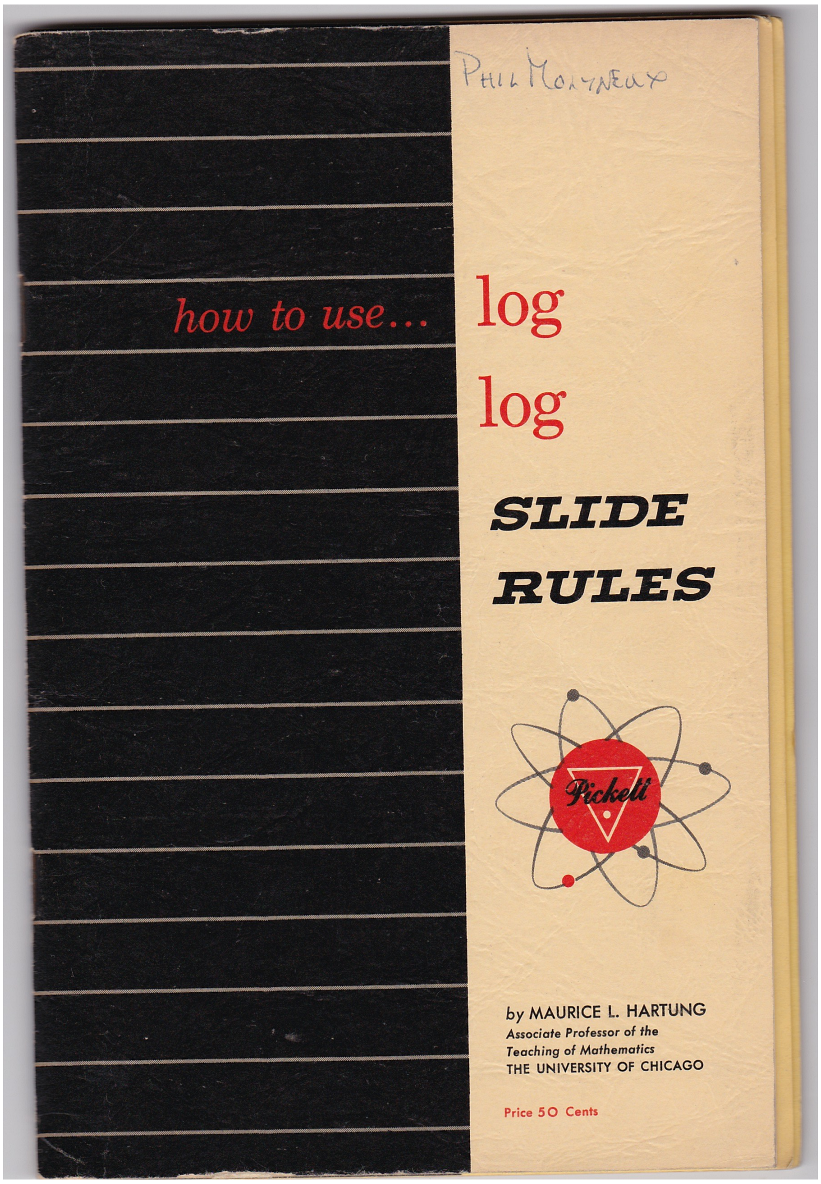
Pickett N 3-ES from 1967





- See Oughtred Society
- UKSRC
- Rod Lovett's Slide Rules
- Slide Rule Museum

Pickett log log **Slide Rules Manual 1953**

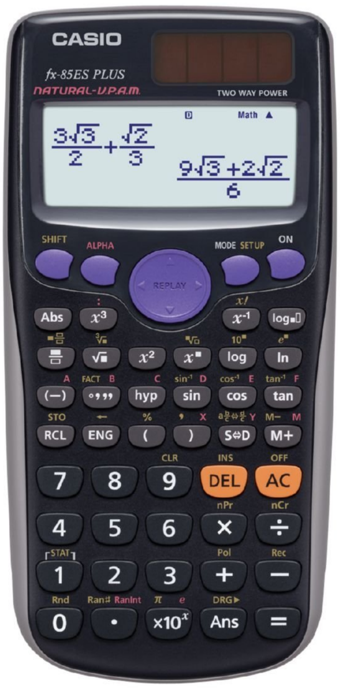


ToC

8.3 Calculators

HP HP-21 Calculator from 1975 £69

Casio fx-85GT PLUS Calculator from 2013 £10



Calculator links

- **HP Calculator Museum** http://www.hpmuseum.org

- **HP Calculator Emulators** http://nonpareil.brouhaha.com
- **HP Calculator Emulators for OS X** http://www.bartosiak.org/nonpareil/
- **Vintage Calculators Web Museum** http://www.vintagecalculators.com

ToC

8.4 Example Calculation

- Evaluate 89.7×597
- **Knott's Tables**
- $\log_{10} 89.7 = 1.9528$ and $\log_{10} 597 = 2.7760$
- Shows *mantissa* (decimal) & *characteristic* (integral)
- Add 4.7288, take *antilog* to get $5346 + 10 = 5.356 \times 10^4$
- **HP-21 Calculator** — set display to 4 decimal places
- 89.7 log = 1.9528 and 597 log = 2.7760
- + displays 4.7288
- 10 ENTER, $x \rightleftarrows y$ and y^x displays 53550.9000
- **Casio fx-85GT PLUS**
- log 89.7) = 1.952792443 + log 597) = 2.775974331 =
- 4.728766774 Ans + 10^x gives 53550.9

ToC

9 Logic and Truth Tables

9.1 Boolean Expressions and Truth Tables

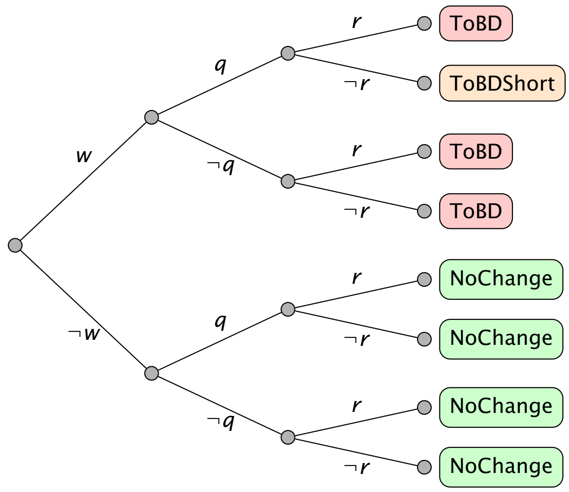
Traffic Lights Example

- Consider traffic light at the intersection of roads *AC* and *BD* with the following rules for the *AC* controller
- Vehicles should not wait on red on *BD* for too long.
- If there is a long queue on *AC* then *BD* is only given a green for a short interval.
- If both queues are long the usual flow times are used.
- We use the following propositions:
  - *w* Vehicles have been waiting on red on *BD* for too long
  - *q* Queue on *AC* is too long
  - *r* Queue on *BD* is too long
- Given the following events:
  - ToBD Change flow to *BD*

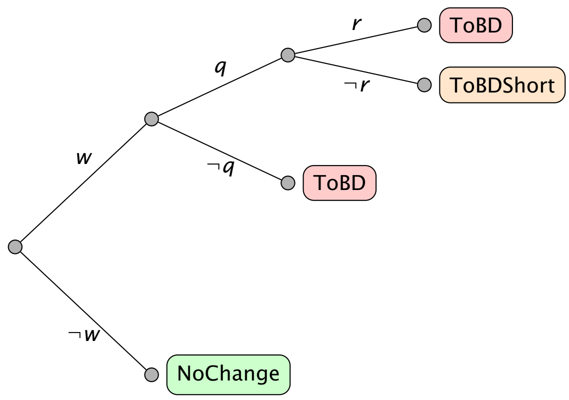
  - ToBDShort Change flow to *BD* for short time
  - NoChange No Change to lights
- Express above as truth table, outcome tree, boolean expression
- **Traffic Lights outcome table**

w	*q*	*r*	Event
T	T	T	ToBD
T	T	F	ToBDShort
T	F	T	ToBD
T	F	F	ToBD
F	T	T	NoChange
F	T	F	NoChange
F	F	T	NoChange
F	F	F	NoChange

- **Traffic lights outcome tree**



- **Traffic lights outcome tree** simplified



- **Traffic Lights code 01**
- See M269TutorialProgPythonADT01.py

```python
def trafficLights01(w,q,r) :
  """
  Input 3 Booleans
  Return Event string
  """
  if w :
```

```
    if q :
      if r :
        evnt = "ToBD"
      else :
        evnt = "ToBDShort"
    else :
      evnt = "ToBD"
  else :
    evnt = "NoChange"
  return evnt
```

- **Traffic Lights test code 01**

```
trafficLights01Evnts = [((w,q,r), trafficLights01(w,q,r))
                              for w in [True,False]
                              for q in [True,False]
                              for r in [True,False]]

assert trafficLights01Evnts \
  == [((True, True, True), 'ToBD')
       ,((True, True, False), 'ToBDShort')
       ,((True, False, True), 'ToBD')
       ,((True, False, False), 'ToBD')
       ,((False, True, True), 'NoChange')
       ,((False, True, False), 'NoChange')
       ,((False, False, True), 'NoChange')
       ,((False, False, False), 'NoChange')]
```

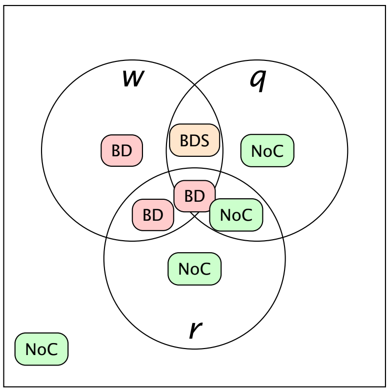
- **Traffic Lights code 02** compound Boolean conditions

```
def trafficLights02(w,q,r) :
  """
  Input 3 Booleans
  Return Event string
  """
  if ((w and q and r) or (w and not q)) :
    evnt = "ToBD"
  elif (w and q and not r) :
    evnt = "ToBDShort"
  else :
    evnt = "NoChange"
  return evnt
```

- What objectives do we have for our code ?
- **Traffic Lights test code 02**

```
trafficLights02Evnts = [((w,q,r), trafficLights02(w,q,r))
                              for w in [True,False]
                              for q in [True,False]
                              for r in [True,False]]

assert trafficLights02Evnts == trafficLights01Evnts
```



- **Traffic Lights Venn diagram**

- OK using a fill colour would look better but didn't have the time to hack the package

ToC

9.2 Conditional Expressions and Validity

- **Validity** of Boolean expressions
- **Complete** every outcome returns an event (or error message, raises an exception)
- **Consistent** — we do not want two nested `if` statements or expressions resulting in different events
- We check this by ensuring that the events form a disjoint partition of the set of outcomes — see the Venn diagram
- We would quite like the programming language processor to warn us otherwise — not always possible

ToC

9.3 Boolean Expressions Exercise

Rail Ticket Exercise

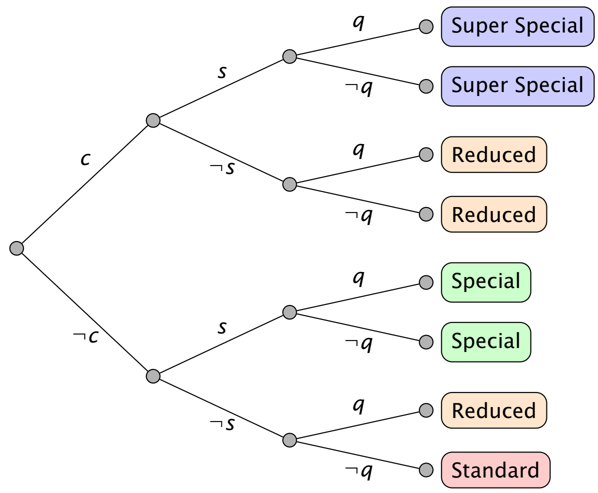
- Rail ticket discounts for:
  - c Rail card
  - q Off-peak time
  - s Special offer
- 4 fares: Standard, Reduced, Special, Super Special
- Rules:
  1. Reduced fare if rail card or at off-peak time
  2. Without rail card no reduction for both special offer and off-peak.
  3. Rail card always has reduced fare but cannot get off-peak discount as well.
  4. Rail card gets super special discount for journey with special offer
- Draw up truth table, outcome tree, Venn diagram and conditional statement (or expression) for this
- Rail ticket outcome table

c	q	s	Event
T	T	T	Super Special
T	T	F	Reduced
T	F	T	Super Special
T	F	F	Reduced
F	T	T	Special
F	T	F	Reduced
F	F	T	Special
F	F	F	Standard

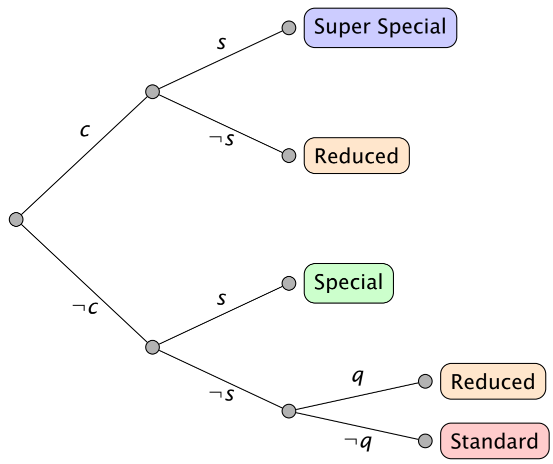
- Rail ticket outcome table
- Note that it may be more convenient to change columns

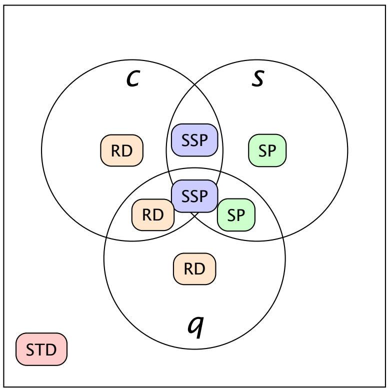
c	s	q	Event
T	T	T	Super Special
T	T	F	Super Special
T	F	T	Reduced
T	F	F	Reduced
F	T	T	Special
F	T	F	Special
F	F	T	Reduced
F	F	F	Standard

- Real fares are a little more complex — see brfares.com
- **Rail Ticket outcome tree**



- **Rail Ticket outcome tree** simplified





- **Rail Ticket Venn diagram**
- **Rail Ticket code 01**

```python
def railTicket01(c,s,q) :
  """
  Input 3 Booleans
  Return Event string
  """
  if c :
    if s :
      evnt = "SSP"
    else :
      evnt = "RD"
  else :
    if s :
      evnt = "SP"
    else :
      if q :
        evnt = "RD"
      else :
        evnt = "STD"
  return evnt
```

- **Rail Ticket test code 01**

```python
railTicket01Evnts = [((c,s,q), railTicket01(c,s,q))
                              for c in [True,False]
                              for s in [True,False]
                              for q in [True,False]]

assert railTicket01Evnts \
  == [((True, True, True), 'SSP')
       ,((True, True, False), 'SSP')
       ,((True, False, True), 'RD')
       ,((True, False, False), 'RD')
       ,((False, True, True), 'SP')
       ,((False, True, False), 'SP')
       ,((False, False, True), 'RD')
       ,((False, False, False), 'STD')]
```

- **Rail Ticket code 02** compound Boolean expressions

```python
def railTicket02(c,s,q) :
  """
  Input 3 Booleans
  Return Event string
  """
  if (c and s) :
    evnt = "SSP"
  elif ((c and not s) or (not c and not s and q)) :
    evnt = "RD"
  elif (not c and s) :
    evnt = "SP"
  else :
    evnt = "STD"
  return evnt
```

- **Rail Ticket test code 02**

```
railTicket02Evnts = [((c,s,q), railTicket02(c,s,q))
                     for c in [True,False]
                     for s in [True,False]
                     for q in [True,False]]

assert railTicket02Evnts == railTicket01Evnts
```

ToC

9.4 Propositional Calculus

- Unit 2 section 3.2 *A taste of formal logic* introduces Propositional calculus
- A language for calculating about Booleans — *truth values*
- Gives operators (connectives) conjunction ($\wedge$) AND, disjunction ($\vee$) OR, negation ($\neg$) NOT, implication ($\Rightarrow$) IF
- There are 16 possible functions $(\mathbb{B}, \mathbb{B}) \to \mathbb{B}$ — see below — defined by their truth tables
- **Discussion** Did you find the truth table for implication weird or surprising ?
- **Implication** has a **negative** definition — we accept its truth unless we have contrary evidence
- $T \Rightarrow T == T$ and $T \Rightarrow F == F$
- Hence 4 possibilities for truth table

p	q	$p \Rightarrow q$	q	$p \Leftrightarrow q$	$p \wedge q$
T	T	T	T	T	T
T	F	F	F	F	F
F	T	T	T	F	F
F	F	T	F	T	F

- ($\Rightarrow$) must have the entry shown — the others are taken
- Do not think of p *causing* q
- **Functionally complete** set of connectives is one which can be used to express all possible connectives
- $p \Rightarrow q \equiv \neg p \vee q$ so we could just use $\{\neg, \wedge, \vee\}$
- **Boolean programming** — we have to have a functionally complete set but choose more to make the programming easier
- *Expressiveness* is an issue in programming language design
- **NAND** $p \overline{\wedge} q$, $p \uparrow q$, Sheffer stroke
- **NOR** $p \overline{\vee} q$, $p \downarrow q$, Pierce's arrow
- See truth tables below — both $\{\uparrow\}, \{\downarrow\}$ are functionally complete
- **Exercise** verify

- $\neg p \equiv p \uparrow p$
- $p \wedge q \equiv \neg(p \uparrow q) = (p \uparrow q) \uparrow (p \uparrow q)$
- $p \vee q \equiv (p \uparrow p) \uparrow (q \uparrow q)$
- $\neg p \equiv p \downarrow p$
- $p \wedge q \equiv (p \downarrow p) \downarrow (q \downarrow q)$
- $p \vee q \equiv \neg(p \downarrow q) = (p \downarrow q) \downarrow (p \downarrow q)$

- Not a novelty — the Apollo Guidance Computer was implemented in NOR gates alone.

ToC

9.5 Truth Function

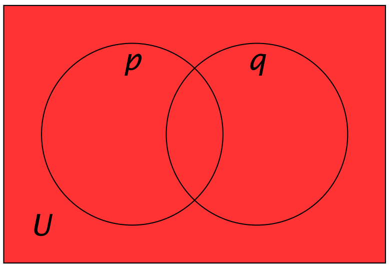
- The following appendix notes illustrate the 16 binary functions of two Boolean variables
- See Truth function
- See Functional completeness
- See Sheffer stroke
- See Logical NOR

Table of Binary Truth Functions

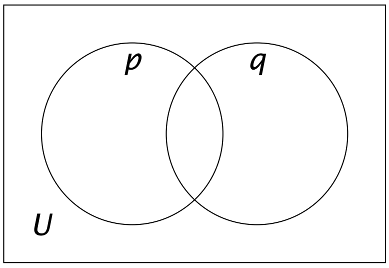
p	q	$\top$	$p \vee q$	$p \Leftarrow q$	p	$p \Rightarrow q$	q	$p \Leftrightarrow q$	$p \wedge q$
T	T	T	T	T	T	T	T	T	T
T	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F

p	q	$\bot$	$p \overline{\vee} q$	$p \nLeftarrow q$	$\neg p$	$p \nRightarrow q$	$\neg q$	$p \nLeftrightarrow q$	$p \overline{\wedge} q$
T	T	F	F	F	F	F	F	F	F
T	F	F	F	F	F	T	T	T	T
F	T	F	F	T	T	F	F	T	T
F	F	F	T	F	T	F	T	F	T

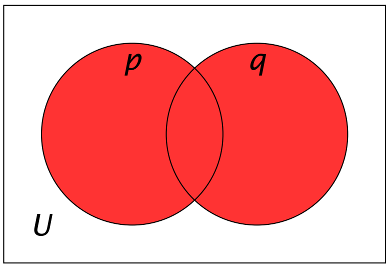
- **Tautology** True, $\top$, *Top*



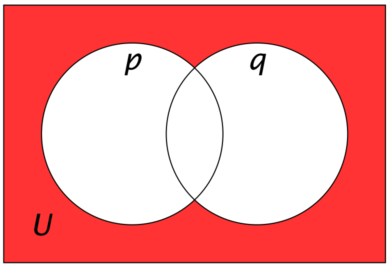
- **Contradiction** False, $\perp$, *Bottom*



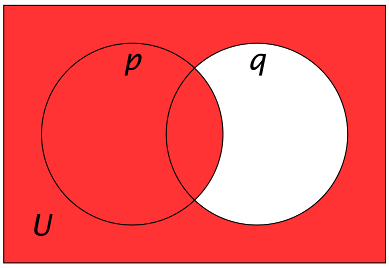
- **Disjunction** OR, $p \vee q$



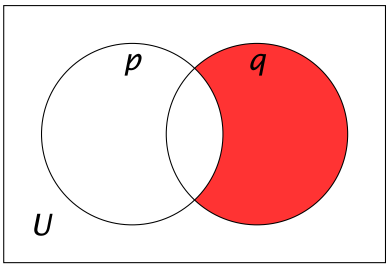
- **Joint Denial** NOR, $p \overline{\vee} q$, $p \downarrow q$, *Pierce's arrow*



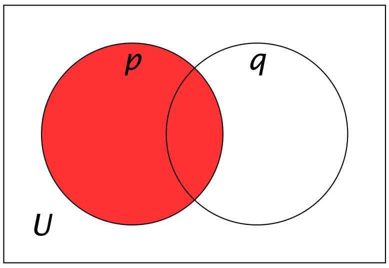
- **Converse Implication** $p \Leftarrow q$



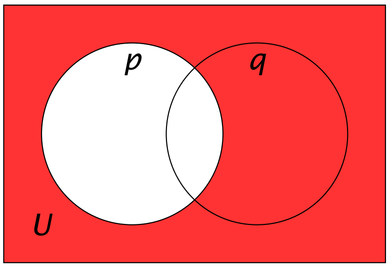
- **Converse Nonimplication** $p \nLeftarrow q$



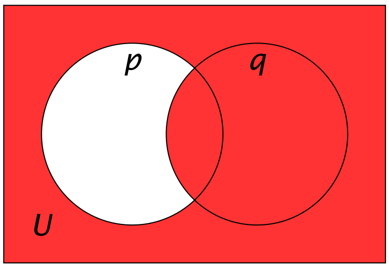
- **Proposition** p



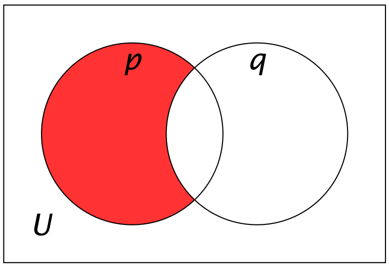
- **Negation of** p



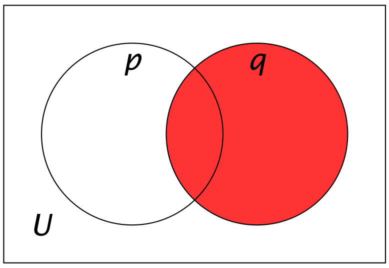
- **Material Implication** $p \Rightarrow q$



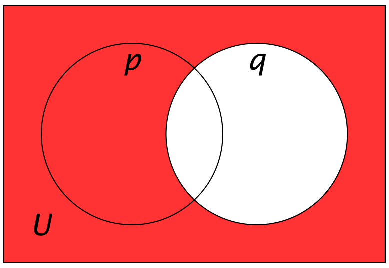
- **Material Nonimplication** $p \nRightarrow q$



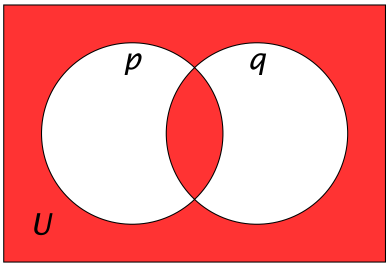
- **Proposition** q q



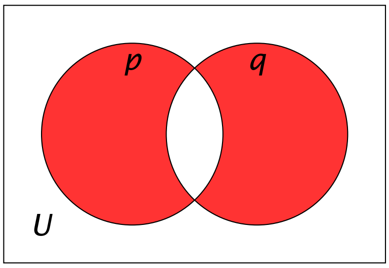
- **Negation of** q $\neg q$



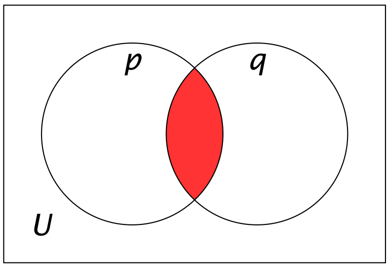
- **Biconditional** If and only if, IFF, $p \Leftrightarrow q$



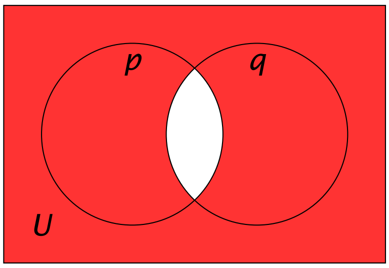
- **Exclusive disjunction** XOR, $p \nLeftrightarrow q$



- **Conjunction** AND, $p \wedge q$



- **Alternative denial** NAND, $p \barwedge q$, $p \uparrow q$, *Sheffer stroke*



ToC

10 Web Sites & References

10.1 Web Sites

- **Logic**

  - WFF, WFF'N Proof online
- **Computability**
  - Computability
  - Computable function
  - Decidability (logic)
  - Turing Machines
  - Universal Turing Machine
  - Turing machine simulator
  - Lambda Calculus
  - Von Neumann Architecture
  - Turing Machine XKCD 205 Candy Button Paper
  - Turing Machine XKCD 505 A Bunch of Rocks
  - RIP John Conway Why can Conway's Game of Life be classified as a universal machine?
  - Phil Wadler Bright Club on Computability
  - Bridges: Theory of Computation: Halting Problem
  - Bridges: Theory of Computation: Other Non-computable Problems
- **Complexity**
  - Complexity class
  - NP complexity
  - NP complete
  - Reduction (complexity)
  - P versus NP problem
  - Graph of NP-Complete Problems

Go to Table of Contents

Note on References — the list of references is mainly to remind me where I obtained some of the material and is not required reading.

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