Binary Trees

M269 Tutorial

Phil Molyneux

12 January 2025

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Agenda, Aims and Topics

1 Agenda, Aims and Topics

- Overview of aims of tutorial
- Note selection of topics
- Recursion is used throughout the topics
- Points about my own background and preferences
- Adobe Connect slides for reference

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Agenda & Aims

1. Welcome and introductions
2. Material on Binary Trees and Searching
3. Implementation in Python
4. Learning themes:
   - Evaluation of expressions
   - Synthesising an algorithm from an initial idea
5. To cover some of
   - Binary Trees
   - Binary Search Trees
   - Height Balanced (AVL) Trees
6. Questions & discussion (at any point)
7. *Adobe Connect* — if you or I get cut off, wait till we reconnect (or send you an email)
8. *Source:* of slides, notes, programs:

   pmolyneux.co.uk/OU/M269FolderSync/M269TutorialSamples2023/

   M269TutorialBinaryTreesCmntry2023/

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Agenda

- There is a lot more material in these slides/notes than we can cover in the available time, so I will cover:

(1) Binary Tree terminology and representation — some choices

(2) Tree traversal — depth first recursive

(3) Tree traversal — breadth first — recursive first, transformed to the usual iterative version

- These notes are as much about recursion as Binary trees — the notes give several examples of evaluations and what to do when you make a mistake

(4) Binary search trees — deleting a node — choices

(5) AVL or height balanced trees — brief introduction

Health Warning These notes contain some material that is not part of M269 but is present for interest

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Materials

- From the Web link to the folder containing the tutorial materials you should find:
- File with name ending `.beamer.pdf` — the slides
- File with name ending `.article.pdf` — the notes version
- Table of contents — in the slides this is a clickable sidebar; in the notes it is an expanded list of sections with links from the end of sections
- Indices — the notes version has an index of the Python code and the diagrams
- References — the notes version has references which have back references to the pages where the reference is cited

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Introductions — Phil

- *Name* Phil Molyneux
- *Background*
  - Undergraduate: Physics and Maths (Sussex)
  - Postgraduate: Physics (Sussex), Operational Research (Brunel), Computer Science (University College, London)
  - Worked in Operational Research, Business IT, Web technologies, Functional Programming
- *First programming languages* Fortran, BASIC, Pascal
- *Favourite Software*
  - Haskell — pure functional programming language
  - Text editors TextMate, Sublime Text — previously Emacs
  - Word processing in LaTeX — all these slides and notes
  - Mac OS X
- *Learning style* — I read the manual before using the software

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Introductions — You

- *Name ?*
- *Favourite software/Programming language ?*
- *Favourite text editor or integrated development environment (IDE)*
- List of text editors, Comparison of text editors and Comparison of integrated development environments
- *Other OU courses ?*
- *Anything else ?*

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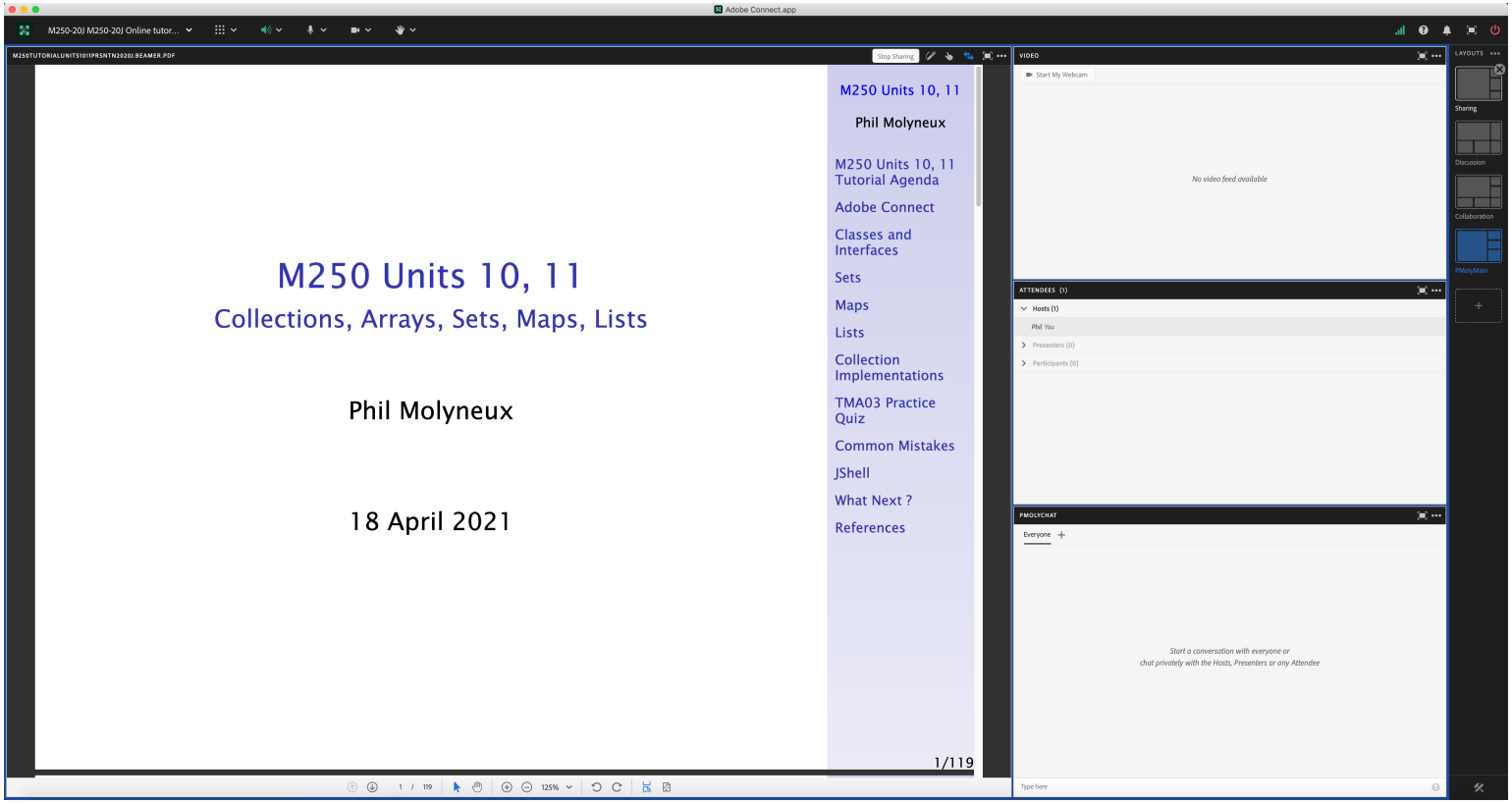
Adobe Connect

Interface — Host View



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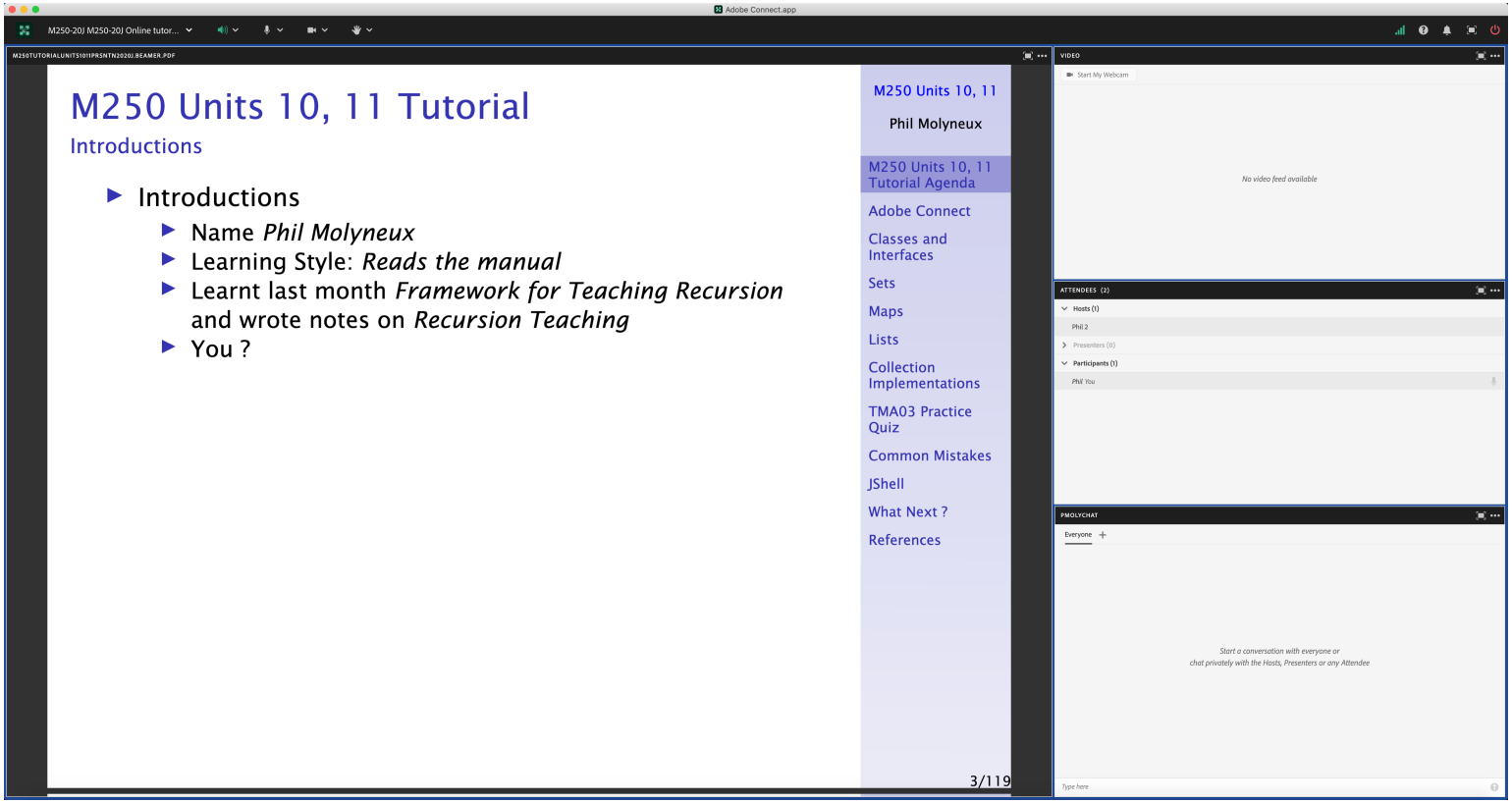
Interface — Participant View



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Adobe Connect

Settings

- **Everybody** *Menu bar* › *Meeting* › Speaker & Microphone Setup
- *Menu bar* › *Microphone* › Allow Participants to Use Microphone ✔
- Check Participants see the entire slide **Workaround**
  - *Disable Draw* *Share pod* › *Menu bar* › *Draw icon*
  - *Fit Width* *Share pod* › *Bottom bar* › *Fit Width icon* ✔
- *Meeting* › Preferences › General › **Host Cursor** › Show to all attendees
- *Menu bar* › *Video* › Enable **Webcam** for Participants ✔
- Do not *Enable single speaker mode*
- Cancel hand tool
- Do not enable green pointer
- **Recording** Meeting › Record Session ✔
- **Documents** Upload PDF with drag and drop to share pod
- Delete *Meeting* › Manage Meeting Information › Uploaded Content and *check filename* › *click on delete*

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Access

- **Tutor Access**

  TutorHome > M269 Website > Tutorials

  Cluster Tutorials > M269 Online tutorial room

  Tutor Groups > M269 Online tutor group room

  Module-wide Tutorials > M269 Online module-wide room

- **Attendance**

  TutorHome > Students > View your tutorial timetables

- **Beamer Slide Scaling** 440% (422 x 563 mm)

- **Clear Everyone's Status**

  Attendee Pod > Menu > Clear Everyone's Status

- **Grant Access** and send link via email

  Meeting > Manage Access & Entry > Invite Participants...

- **Presenter Only Area**

  Meeting > Enable/Disable Presenter Only Area

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Adobe Connect

Keystroke Shortcuts

- Keyboard shortcuts in Adobe Connect
- **Toggle Mic** ⌘ + M (Mac), Ctrl + M (Win) (On/Disconnect)
- **Toggle Raise-Hand status** ⌘ + E
- **Close dialog box** ⎋ (Mac), Esc (Win)
- **End meeting** ⌘ + \

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Adobe Connect Interface

Sharing Screen & Applications

- Share My Screen > Application tab > Terminal for Terminal
- Share menu > Change View > Zoom in for mismatch of screen size/resolution (Participants)
- (Presenter) Change to 75% and back to 100% (solves participants with smaller screen image overlap)
- Leave the application on the original display
- Beware blued hatched rectangles — from other (hidden) windows or contextual menus
- Presenter screen pointer affects viewer display — beware of moving the pointer away from the application
- First time: System Preferences > Security & Privacy > Privacy > Accessibility

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Ending a Meeting

- *Notes for the tutor only*
- **Student:** Meeting > Exit Adobe Connect
- **Tutor:**
- **Recording** Meeting > Stop Recording ✔
- **Remove Participants** Meeting > End Meeting... ✔
  - Dialog box allows for message with default message:
  - *The host has ended this meeting. Thank you for attending.*
- **Recording availability** *In course Web site for joining the room, click on the eye icon in the list of recordings under your recording* — edit description and name
- **Meeting Information** Meeting > Manage Meeting Information — can access a range of information in Web page.
- **Delete File Upload** Meeting > Manage Meeting Information > Uploaded Content tab select file(s) and click Delete
- **Attendance Report** see course Web site for joining room

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Adobe Connect

Invite Attendees

- **Provide Meeting URL** Menu > Meeting > Manage Access & Entry > Invite Participants...
- **Allow Access without Dialog** Menu > Meeting > Manage Meeting Information provides new browser window with *Meeting Information* Tab bar > Edit Information
- Check *Anyone who has the URL for the meeting can enter the room*
- Default *Only registered users and accepted guests may enter the room*
- **Reverts to default next session but URL is fixed**
- Guests have blue icon top, registered participants have yellow icon top — same icon if URL is open
- See Start, attend, and manage Adobe Connect meetings and sessions

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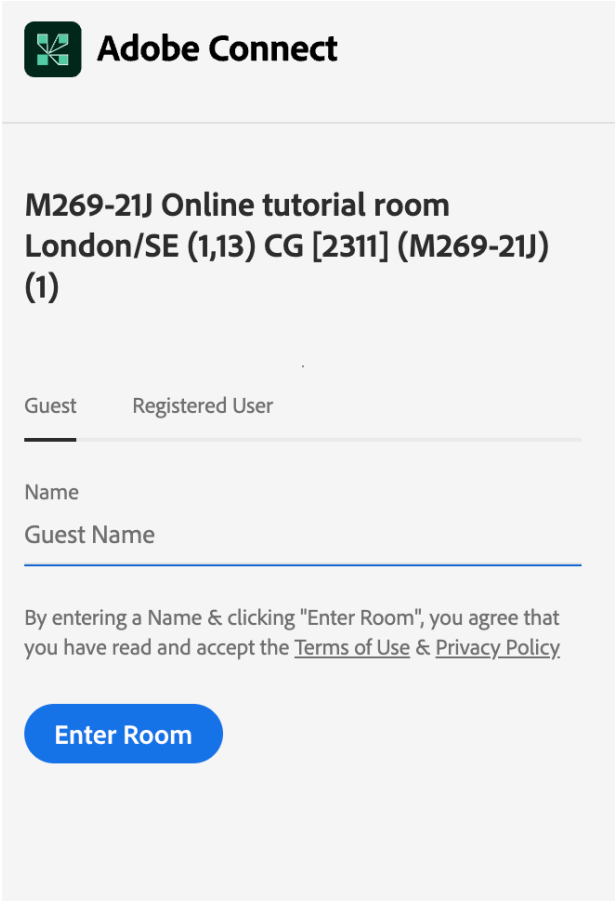
Entering a Room as a Guest (1)

- Click on the link sent in email from the Host
- Get the following on a Web page
- As *Guest* enter your name and click on Enter Room



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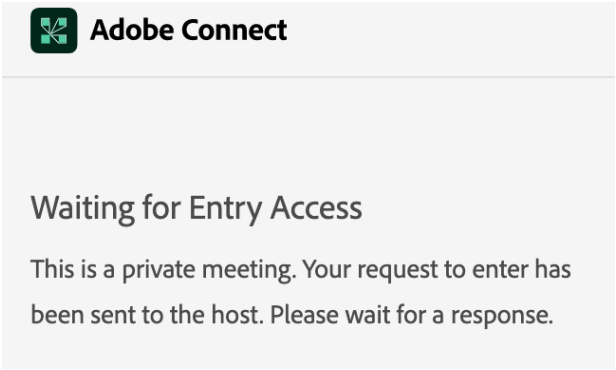
Adobe Connect

Entering a Room as a Guest (2)

- See the *Waiting for Entry Access* for *Host* to give permission



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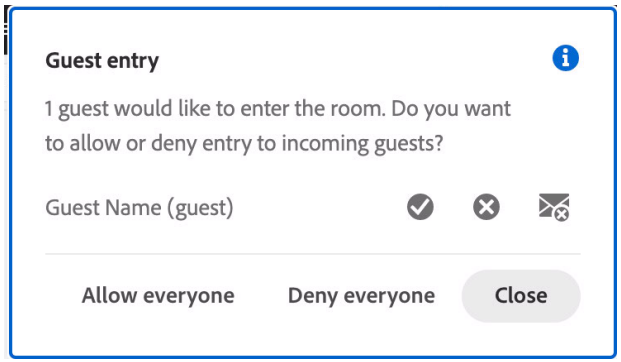
Adobe Connect

Entering a Room as a Guest (3)

- *Host* sees the following dialog in *Adobe Connect* and grants access



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Layouts

- **Creating new layouts** example *Sharing* layout
- Menu > Layouts > Create New Layout... > Create a New Layout dialog > Create a new blank layout and name it *PMolyMain*
- New layout has no Pods but does have Layouts Bar open (see Layouts menu)
- **Pods**
- Menu > Pods > Share > Add New Share and resize/position — initial name is *Share n* — rename *PMolyShare*
- **Rename Pod** Menu > Pods > Manage Pods... > Manage Pods > Select > Rename or Double-click & rename
- Add Video pod and resize/reposition
- Add Attendance pod and resize/reposition
- Add Chat pod — rename it *PMolyChat* — and resize/reposition

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Layouts

- Dimensions of **Sharing** layout (on 27-inch iMac)
  - Width of Video, Attendees, Chat column 14 cm
  - Height of Video pod 9 cm
  - Height of Attendees pod 12 cm
  - Height of Chat pod 8 cm
- **Duplicating Layouts** does *not* give new instances of the Pods and is probably not a good idea (apart from local use to avoid delay in reloading Pods)
- **Auxiliary Layouts** name *PMolyAux0n*
  - Create new Share pod
  - Use existing Chat pod
  - Use same Video and Attendance pods

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Chat Pods

- **Format Chat text**
- Chat Pod > menu icon > My Chat Color
- Choices: Red, Orange, Green, Brown, Purple, Pink, Blue, Black
- Note: Color reverts to Black if you switch layouts
- Chat Pod > menu icon > Show Timestamps

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Graphics Conversion

PDF to PNG/JPG

- Conversion of the screen snaps for the installation of Anaconda on 1 May 2020
- Using GraphicConverter 11
- File > Convert & Modify > Conversion > Convert
- Select files to convert and destination folder
- Click on Start selected Function or ⌘ + ⏎

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Exporting Recordings

- *Menu bar* > *Meeting* > Preferences > Video
- Aspect ratio > Standard (4:3) (not Wide screen (16:9) default)
- Video quality > Full HD (1080p not High default 480p)
- **Recording** *Menu bar* > *Meeting* > Record Session ✔
- **Export Recording**
- *Menu bar* > *Meeting* > Manage Meeting Information
- *New window* > Recordings > *check Tutorial* > Access Type button
- check Public > check Allow viewers to download
- **Download Recording**
- *New window* > Recordings > *check Tutorial* > Actions > Download File

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Binary Trees

2 Binary Trees

- Usage, terminology, example trees
- Representation, Abstract Data Types and notation
- Tree traversals, Depth First and Breadth First
- Recursive versions first
- Iterative versions derived from recursive versions
- Iterative depth first traversals for interest only
- Points on performance

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Binary Trees

Introduction

- The *tree data structure* is the most widely used non-linear structure in many algorithms.
- Almost all algorithms that take logarithmic time, $O(\log n)$, do so because of an underlying tree structure.
- Common examples
- Binary search tree — this is used in many search applications
- Huffman coding tree — used in compression algorithms in, for example, JPEG and MP3 files
- Heaps — used to implement priority queues
- B-trees — generalisation of Binary search trees used in databases.

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Binary Trees

Terminology

- **Binary Tree definition** — a Binary tree is either
  - an `Empty Tree` or
  - a `Node with an item and two subtrees`
  - One subtree is designated a left subtree and the other a right subtree
- Note that this is a recursive or inductive definition — this is very common in programming.
- Can also define trees as *graphs without cycles* — see *graph notes*

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Other Recursive Data Structures

- Other examples of recursive or inductively defined data structures we have seen include:
- A `List` is either
  - an `Empty List` or
  - an `Item followed by the rest of the list`
- A `Stack` is either
  - an `Empty Stack` or
  - the `Top item followed by the rest of the stack`
- In each case the recursive nature of the data structure definition frequently gives a clue about how to write a recursive program for a computational problem.

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Terminology

- `Children` — subtrees of a node that are not empty
- `Leaves` — nodes with two empty subtrees
- `Full Binary Tree` — every node other than the leaves has two non empty subtrees
- `Perfect Binary Tree` — all leaves are at the same level (or depth) children
- `Complete Binary Tree` — every level, except possibly the last, is completely filled, and all nodes are as far left as possible — used for *Binary Heap*
- **Health Warning:** the terminology varies from text to text and between graph theory in mathematics and computing.

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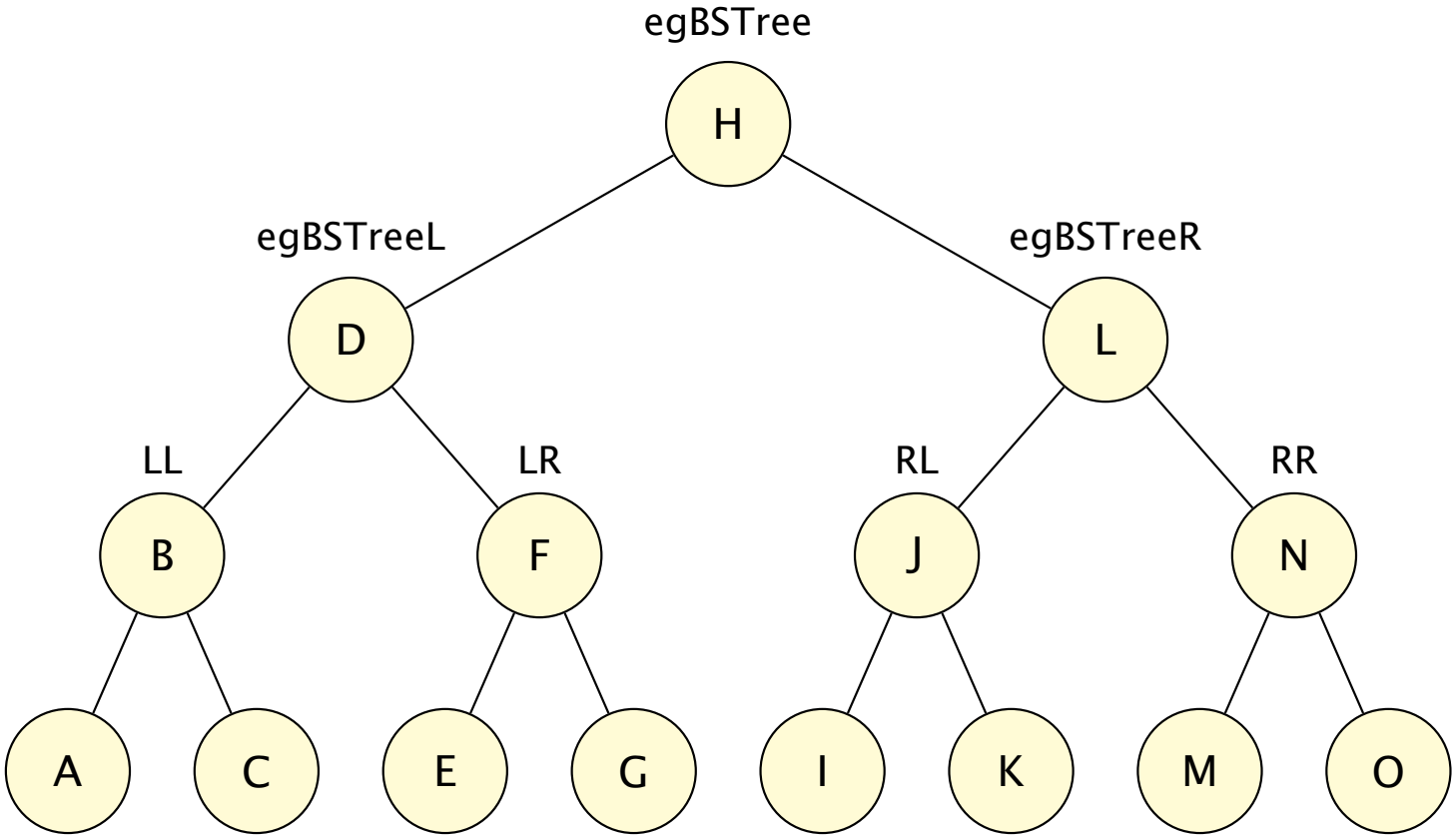
Example egBSTree



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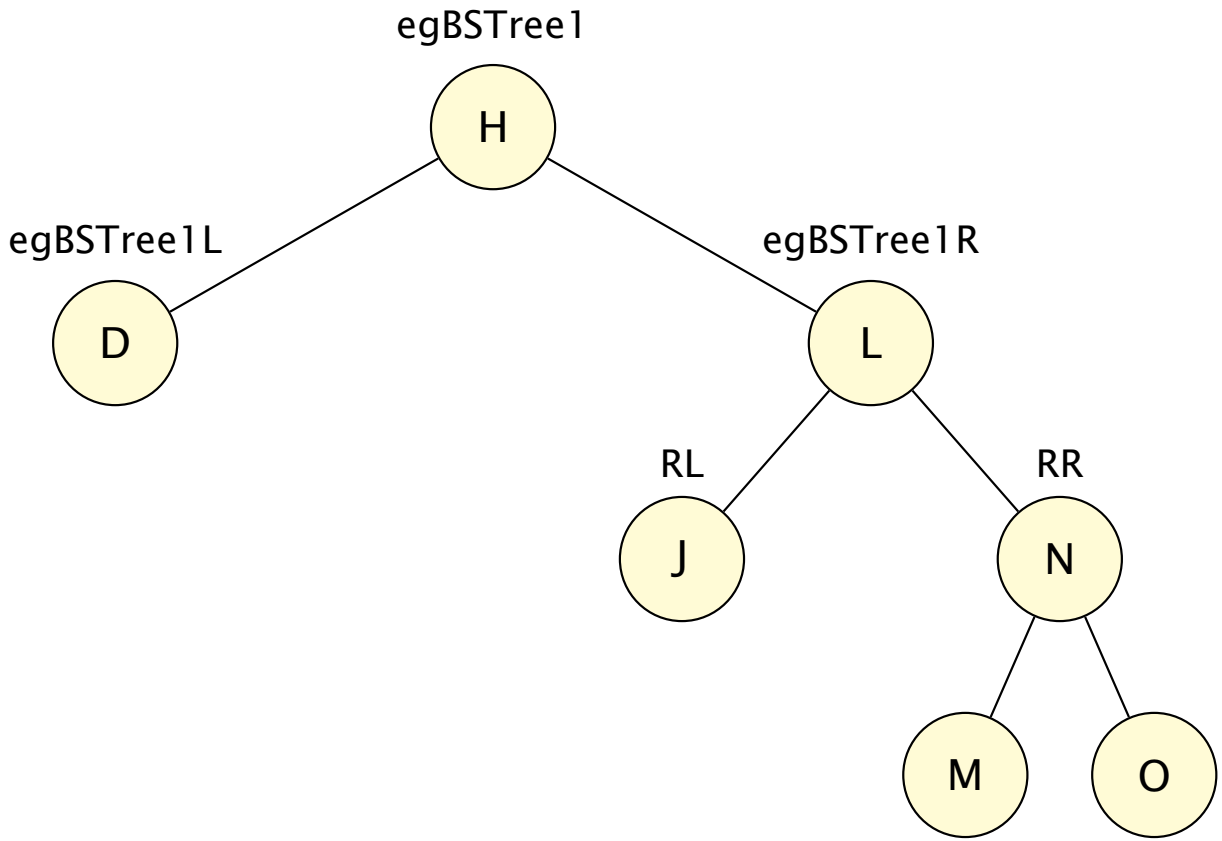
Example egBSTree1



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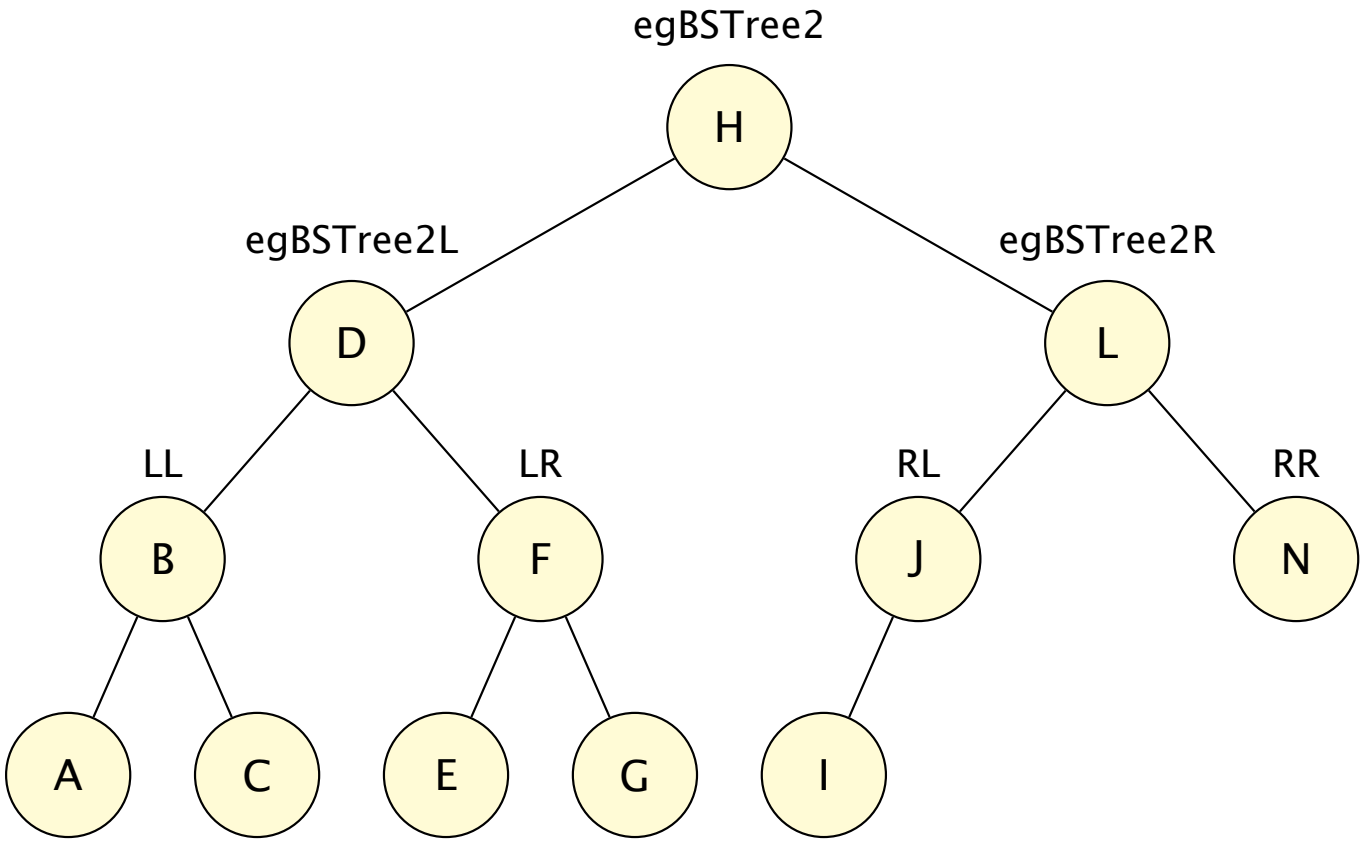
Example egBSTree2



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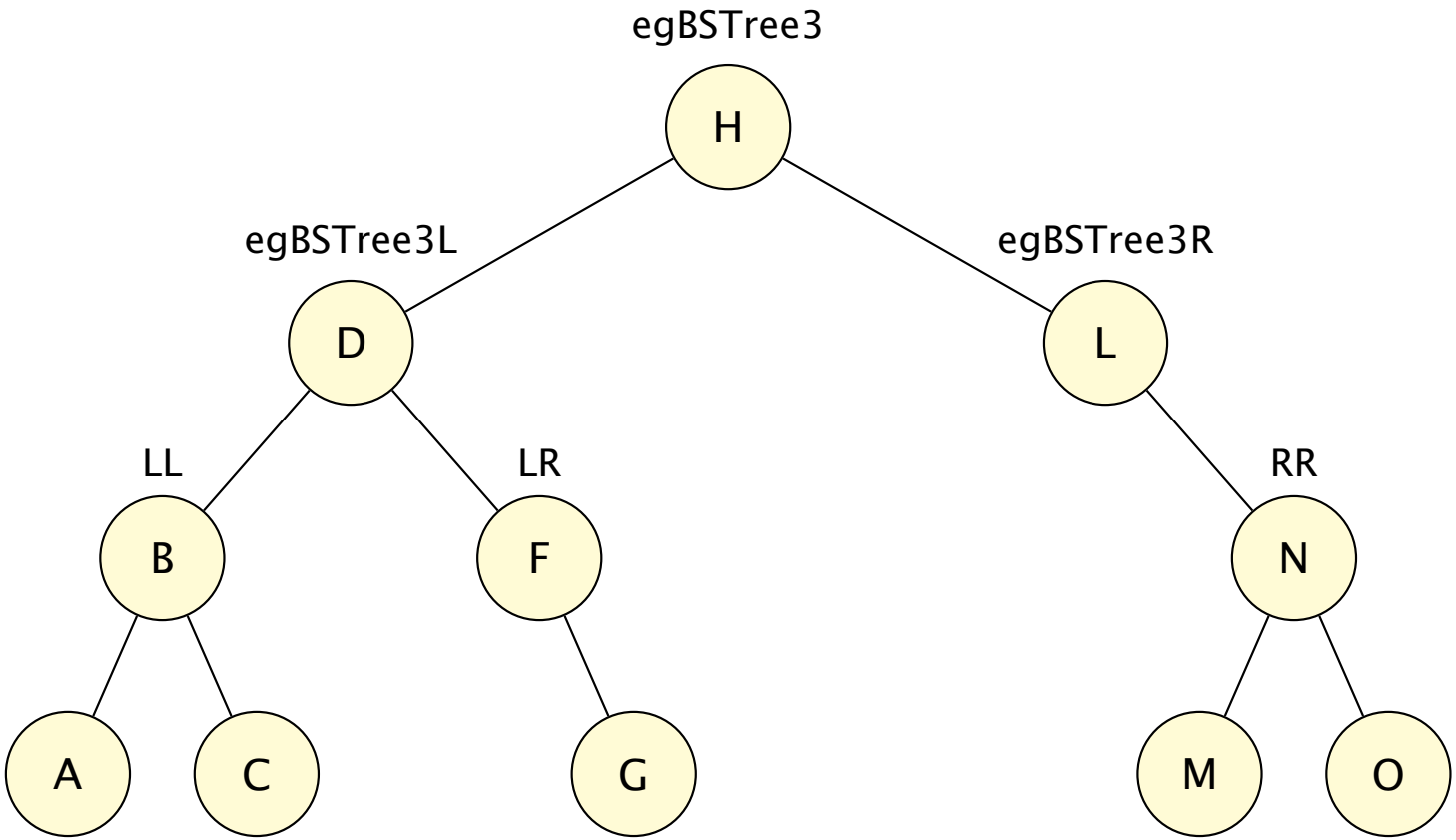
Example egBSTree3



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Binary Trees

Activity 1 Binary Tree Types

- What types of trees are the above example trees ?
- egBSTree
- egBSTree1
- egBSTree2
- egBSTree3

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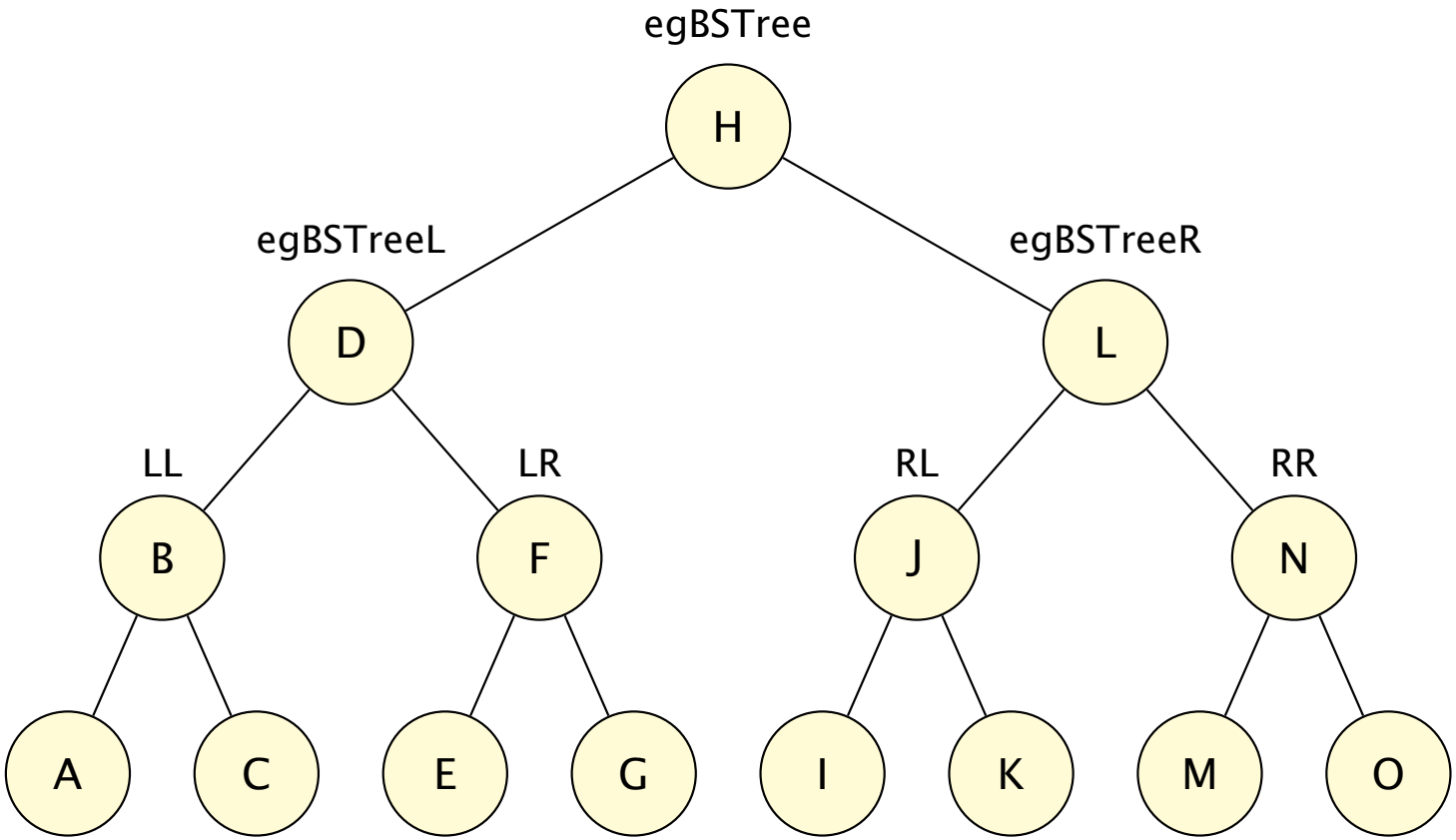
Answer 1 Binary Tree Types (a)

- egBSTree — perfect



- Answer 1 continued on next slide

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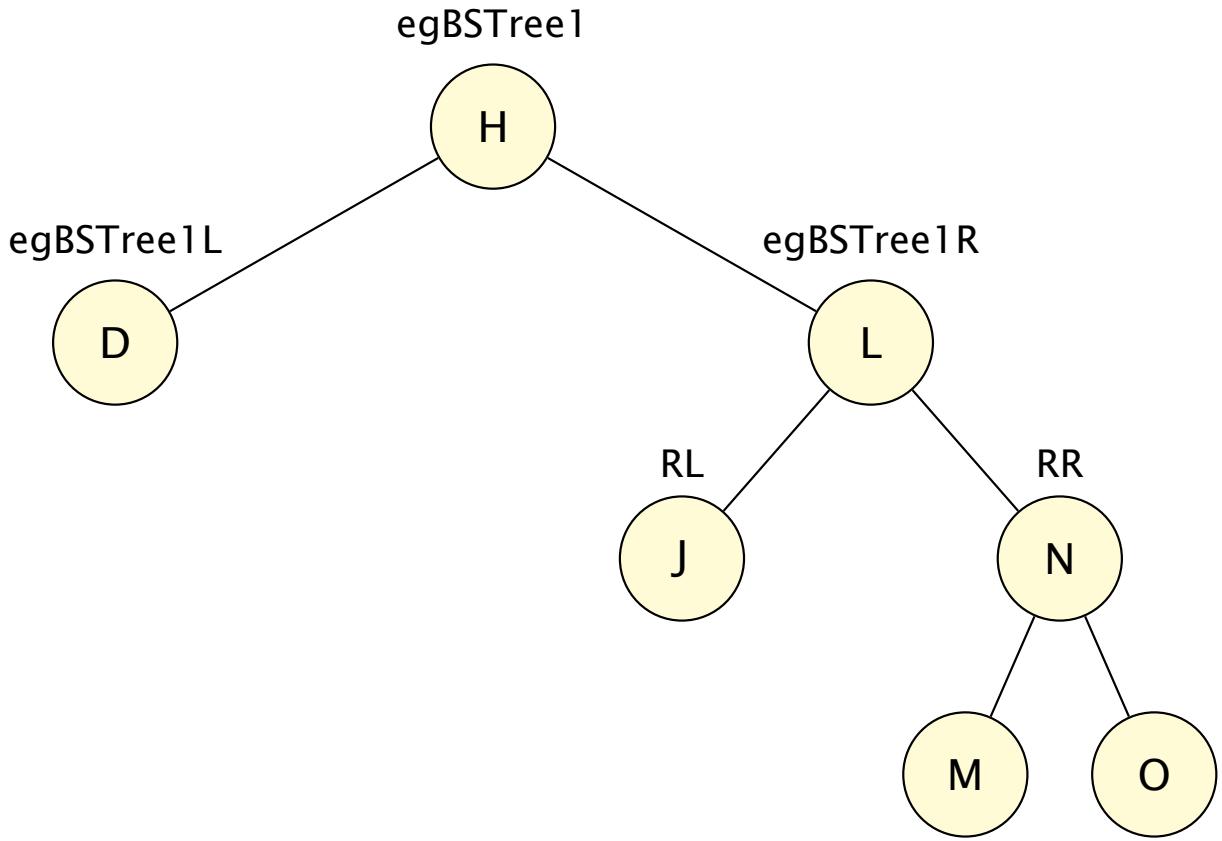
Answer 1 Binary Tree Types (b)

- egBSTree1 — full



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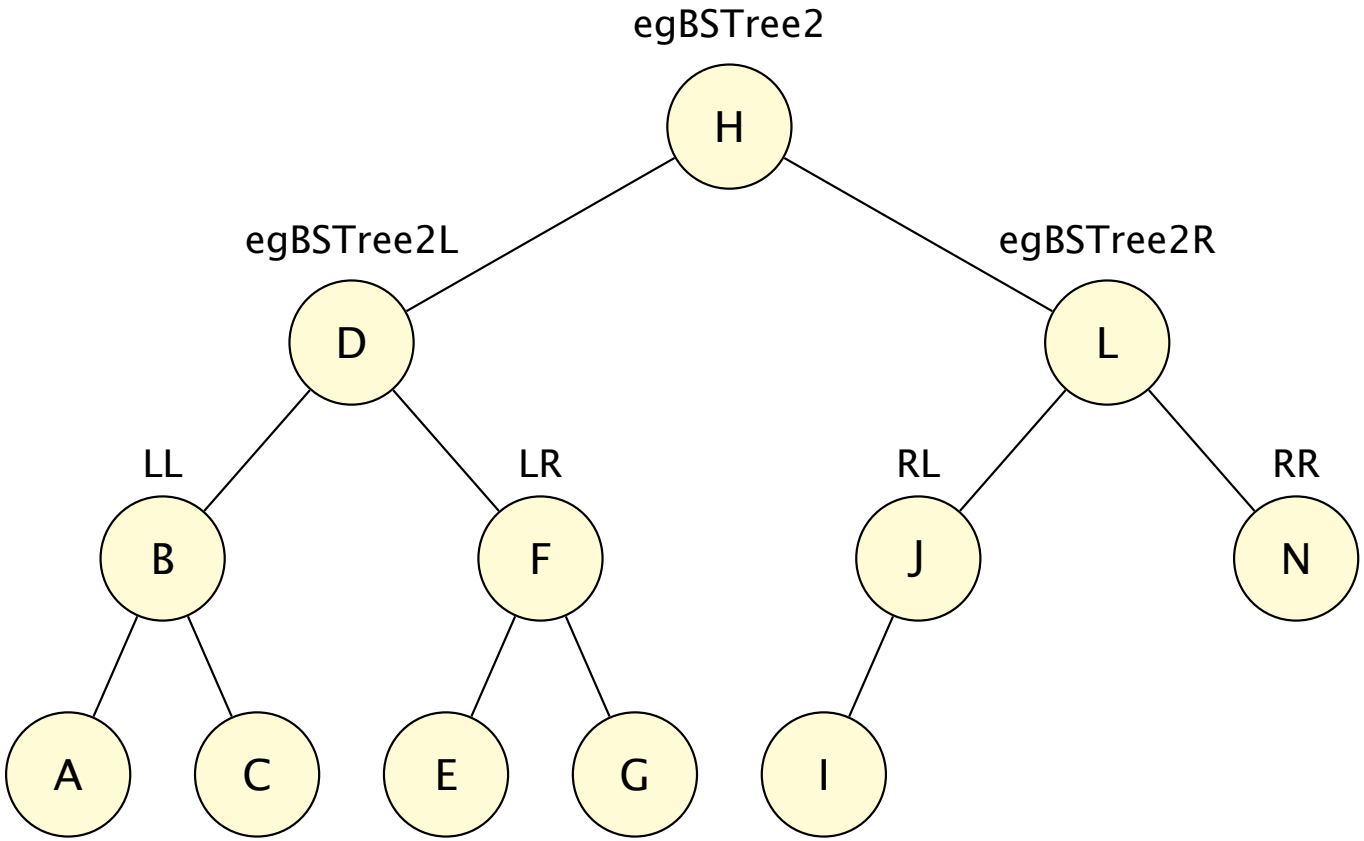
Answer 1 Binary Tree Types (c)

- egBSTree2 — complete



- Answer 1 continued on next slide

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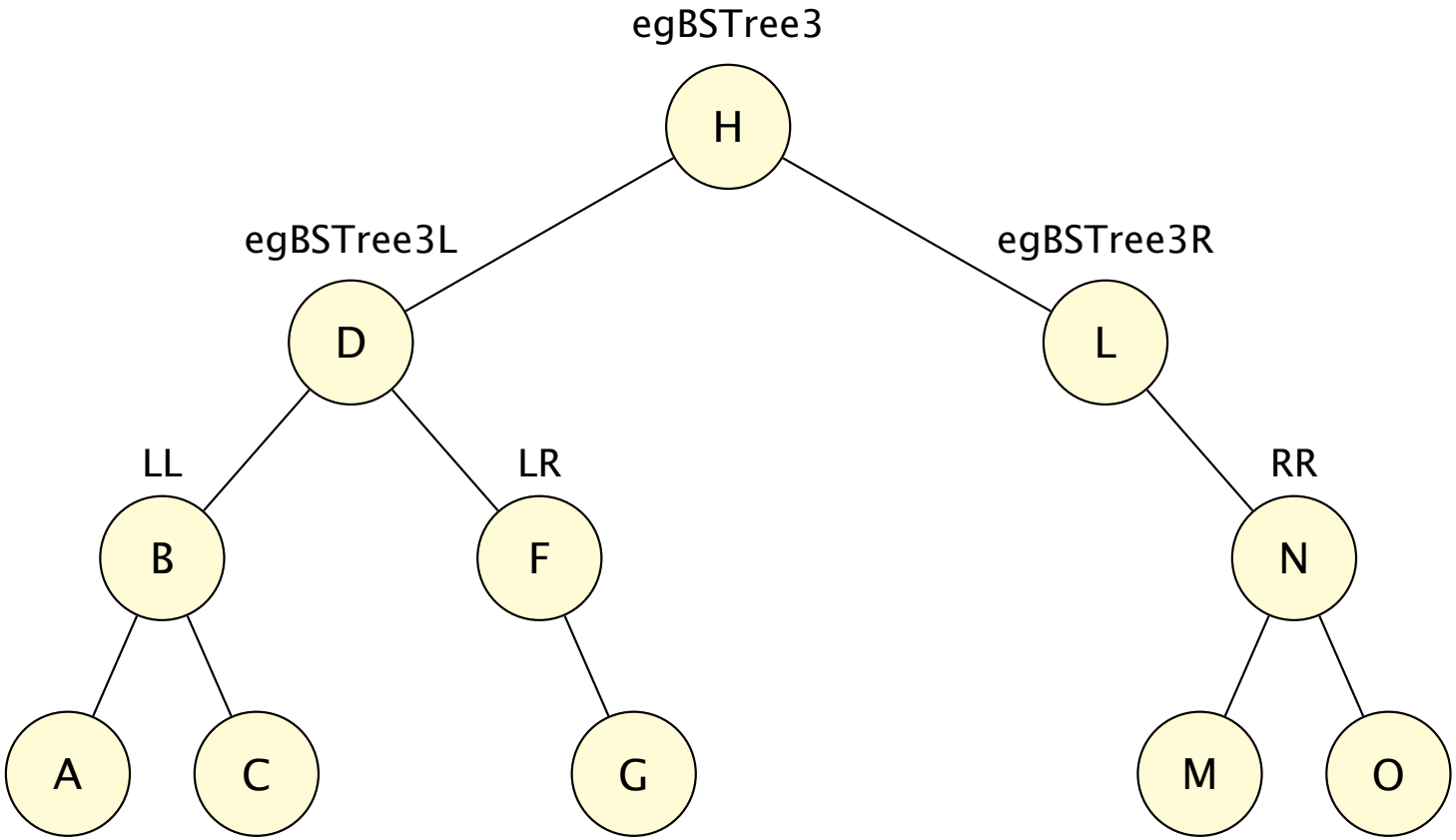
Answer 1 Binary Tree Types (d)

- egBSTree3 — just a binary tree



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Python Representation from 2021J (1)

- In 2021J M269 revision the Binary Tree Abstract Data Type (ADT) is represented by the following Python *Class*
- The code is in the M269 Jupyter Notebooks and the provided file `m269_trees.py`
- The code is reproduced in the file `M269BinaryTrees2021J.py` but, for brevity, without the docstrings

```
10 class Tree :

12   def __init__(self) :
13     self.root = None
14     self.left = None
15     self.right = None

17 def is_empty(tree: Tree) -> bool :
18   return (tree.root == tree.left == tree.right
19           == None)

21 def join(item: object, left: Tree, right: Tree) -> Tree :
22   tree = Tree()
23   tree.root = item
24   tree.left = left
25   tree.right  = right
26   return tree
```


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Python Representation from 2021J (2)

- The functions `is_leaf`, `size`, `height`

```
28 def is_leaf(tree: Tree) -> bool :
29   return (not is_empty(tree)
30           and is_empty(tree.left) and is_empty(tree.right))

32 def size(tree: Tree) -> int :
33   if is_empty(tree) :
34     return 0
35   else :
36     return (size(tree.left) + size(tree.right) + 1)

38 def height(tree: Tree) -> int :
39   if is_empty(tree) :
40     return 0
41   else :
42     return (max(height(tree.left), height(tree.right)) + 1)
```


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Python Representation from 2021J (2)

- This representation works (see the M269 book) but has the slight disadvantage in the it has no default print representation that is useful
- For example, here is what happens when we attempt to print a very small tree — `threeBT` is the same as `THREE` in the chapter

```
Python3>>> threeBT = join(3,Tree(),Tree())
Python3>>> threeBT
<M269BinaryTrees2021J.Tree object at 0x10095a0a0>
Python3>>>
```

- We *could* write a method to implement a print representation of an instance of the class `Tree()` but that might be a lot of code

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Python Representation from 2021J (2)

- The main point of having an Abstract Data Type (ADT) is we can swap out the underlying implementation for another one
- This might be done for efficiency reasons but here we do it to get an underlying type with a default print representation
- All we have to do is keep operations which provide access to the underlying representation
- *This is called a learning opportunity!*

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Python Alternate Representation (1)

- We first list the operations which will (or might) need to have direct access to the underlying representation
- Make an empty tree
- Construct a new tree from an item and two trees
- Query if a given tree is an empty tree
- Given a tree, return the left sub tree
- Given a tree, return the right sub tree
- Find the height of a tree
- Find the size of a tree
- The last two do not *need* access to the underlying representation (we can calculate the size and height with just the other operations) but as we will see later, we might give access for efficiency reasons

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Python Alternate Representation (2)

- To fully hide the ADT implementation we give common function names to the operations

Tree Class	Common Name	Category
`Tree()`	`mkEmptyBT()`	Constructor
`join()`	`mkNodeBT()`	Constructor
`is_empty()`	`isEmptyBT()`	Inspector
`tree.root`	`getDataBT()`	Destructor
`tree.left`	`getLeftBT()`	Destructor
`tree.right`	`getRightBT()`	Destructor
`height()`	`heightBT()`	Operation
`size()`	`sizeBT()`	Operation

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Python Alternate Representation (3)

- The functions labelled Constructor, Inspector, Destructor are operations that have direct access to the underlying representation
- `sizeBT()` and `heightBT()` are just ordinary operations in this version of the Tree ADT but for efficiency reasons they may become Inspectors in a later version

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Python Alternate Representation (4)

- We shall represent nodes by a named tuple — a *quick and dirty* object recommended by Guido van Rossum (author of the Python programming language).
- `namedtuple()` is a factory function for creating tuple subclasses with named fields
- It is imported from the `collections` module.
- It has a default print representation

```
7  from collections import namedtuple

9  EmptyBT = namedtuple('EmptyBT',[])

11 NodeBT = namedtuple('NodeBT'
12                     ,['dataBT','leftBT','rightBT'])
```


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Python Alternate Representation (5)

- The Python code above is in the file Python/M269TutorialBinaryTrees2022.py
- The line numbers in the margin correspond to the line numbers in the file.
- Notational convention:
- Python reserved identifiers are shown in **`this color`**
- Python buit-in functions in `this color`
- User defined data constructors and functions such as `NodeBT` and `EmptyBT` are shown in `that color`
- **Health Warning:** these notes may not be totally consistent with syntax colouring.

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Python Alternate Representation (6)

- We declare the Python type for a *Union* type since a Tree is either an empty tree or a non-empty tree
- This is venturing into some of the areas of Python Type Annotations that feel rather awkward but we shall use them in a simple way
- Remember that the Python interpreter only checks the type annotations for validity but not for correctness — they just have to look like proper types but the processor does *not* check them

```
14  # Tree type

16  from typing import TypeVar,Union,NewType

18  T = TypeVar('T')
19  Tree = NewType('Tree',Union[EmptyBT,NodeBT]).
```


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Python Alternate Representation (7)

- Note that using namedtuple means that all items are assumed to have Any types (see mypy: Named tuples)
- You *could* use NamedTuple which is a typed version of namedtuple but this would be getting a lot more complicated than types as used in M269
- in particular you would get involved in specifying user-defined generic types and forward references (since the Tree data type is recursive)
- We could have avoided `Union` by just having NodeBT and representing an empty tree by the Python None
- This would be isomorphic to the Class version with default printing

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Operations (1)

- We now provide functions to create, inspect and take apart binary trees
- The code with the line numbers is the code for the implementation using namedtuples

```
23  def mkEmptyBT() -> Tree :
24    return EmptyBT()

26  def mkNodeBT(x : T,t1 : Tree,t2 : Tree) -> Tree :
27    return NodeBT(x,t1,t2)

29  def isEmptyBT(t : Tree) -> bool:
30    return t == EmptyBT()
```

- `mkEmptyBT`, `mkNodeBT` are *constructor* functions — we could have used the raw named tuples but the discipline is good for you and it makes it easier to refactor in future
- `isEmptyBT` uses the `==` operator for identity check (not identity (`is`))

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Operations (2)

- The code with no line numbers illustrates how the previous implementation using Class Tree can be given the same operations interface

```
def mkEmptyBT() -> Tree :
  return Tree()

def mkNodeBT(x : T,t1 : Tree,t2 : Tree) -> Tree :
  return join(x,t1,t2)

def isEmptyBT(t : Tree) -> bool:
  return is_empty(t)
```


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Operations (3)

- Here are the operations that access the parts of the tree

```
32  def getDataBT(t : Tree) -> T:
33    if isEmptyBT(t):
34      raise RuntimeError("getDataBT applied to EmptyBT()")
35    else:
36      return t.dataBT

38  def getLeftBT(t : Tree) -> Tree :
39    if isEmptyBT(t):
40      raise RuntimeError("getLeftBT applied to EmptyBT()")
41    else:
42      return t.leftBT

44  def getRightBT(t : Tree) -> Tree :
45    if isEmptyBT(t):
46      raise RuntimeError("getRightBT applied to EmptyBT()")
47    else:
48      return t.rightBT
```


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Operations (4)

- The Class Tree implementation of the above operations

```
def getDataBT(t : Tree) -> T:
  if isEmptyBT(t):
    raise RuntimeError("getDataBT applied to empty tree")
  else:
    return t.root

def getLeftBT(t : Tree) -> Tree :
  if isEmptyBT(t):
    raise RuntimeError("getLeftBT applied to empty tree")
  else:
    return t.left

def getRightBT(t : Tree) -> Tree :
  if isEmptyBT(t):
    raise RuntimeError("getRightBT applied to empty tree")
  else:
    return t.right
```


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Operations (5)

- Here are the operations `heightBT()` and `sizeBT()`
- Note that `height` of an empty tree is 0

```
59 def heightBT(t : Tree) -> int :
60   if isEmptyBT(t):
61     return 0
62   else:
63     return (1 + max(heightBT(getLeftBT(t))
64                     ,heightBT(getRightBT(t))))

66 def sizeBT(t : Tree) -> int :
67   if isEmptyBT(t) :
68     return 0
69   else :
70     return (1 + sizeBT(getLeftBT(t))
71               + sizeBT(getRightBT(t)))
```

- The Class Tree implementation of the above operations is exactly the same
- If we make `height` or `size` directly part of the data structure this may change

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Activity 2 Python Representation

- Write Python implementations of the following trees (from the diagrams above) using the named tuple `NodeBT` and `EmptyBT`
- egBSTree
- egBSTree1
- egBSTree2
- egBSTree3

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Answer 2 Python Representation — egBSTree

```
70  egBSTree = mkNodeBT('H',
71                mkNodeBT('D',
72                  mkNodeBT('B',
73                    mkNodeBT('A',mkEmptyBT(),mkEmptyBT()),
74                    mkNodeBT('C',mkEmptyBT(),mkEmptyBT())
75                  ),
76                  mkNodeBT('F',
77                    mkNodeBT('E',mkEmptyBT(),mkEmptyBT()),
78                    mkNodeBT('G',mkEmptyBT(),mkEmptyBT())
79                  )
80                ),
81                mkNodeBT('L',
82                  mkNodeBT('J',
83                    mkNodeBT('I',mkEmptyBT(),mkEmptyBT()),
84                    mkNodeBT('K',mkEmptyBT(),mkEmptyBT())
85                  ),
86                  mkNodeBT('N',
87                    mkNodeBT('M',mkEmptyBT(),mkEmptyBT()),
88                    mkNodeBT('O',mkEmptyBT(),mkEmptyBT())
89                  )
90                )
91              )
```

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Answer 2 Python Representation — egBSTree1

```
195 egBSTree1 = mkNodeBT('H',
196                 mkNodeBT('D',mkEmptyBT(),mkEmptyBT()),
197                 mkNodeBT('L',
198                   mkNodeBT('J',mkEmptyBT(),mkEmptyBT()),
199                   mkNodeBT('N',
200                     mkNodeBT('M',mkEmptyBT(),mkEmptyBT()),
201                     mkNodeBT('O',mkEmptyBT(),mkEmptyBT())
202                   )
203                 )
204               )
```

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Answer 2 Python Representation — egBSTree2

```
221 egBSTree2 = mkNodeBT('H',
222                 mkNodeBT('D',
223                   mkNodeBT('B',
224                     mkNodeBT('A',mkEmptyBT(),mkEmptyBT()),
225                     mkNodeBT('C',mkEmptyBT(),mkEmptyBT())
226                   ),
227                   mkNodeBT('F',
228                     mkNodeBT('E',mkEmptyBT(),mkEmptyBT()),
229                     mkNodeBT('G',mkEmptyBT(),mkEmptyBT())
230                   )
231                 ),
232                 mkNodeBT('L',
233                   mkNodeBT('J',
234                     mkNodeBT('I',mkEmptyBT(),mkEmptyBT()),
235                     mkEmptyBT()
236                   ),
237                   mkNodeBT('N',mkEmptyBT(),mkEmptyBT())
238                 )
239               )
```

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Binary Trees

Answer 2 Python Representation — egBSTree3

```
265 egBSTree3 = mkNodeBT('H',
266                 mkNodeBT('D',
267                   mkNodeBT('B',
268                     mkNodeBT('A',mkEmptyBT(),mkEmptyBT()),
269                     mkNodeBT('C',mkEmptyBT(),mkEmptyBT())
270                   ),
271                   mkNodeBT('F',
272                     mkEmptyBT(),
273                     mkNodeBT('G',mkEmptyBT(),mkEmptyBT())
274                   )
275                 ),
276                 mkNodeBT('L',
277                   mkEmptyBT(),
278                   mkNodeBT('N',
279                     mkNodeBT('M',mkEmptyBT(),mkEmptyBT()),
280                     mkNodeBT('O',mkEmptyBT(),mkEmptyBT())
281                   )
282                 )
283               )
```

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Binary Trees

Answer 2 Python Representation — egBSTreeL

```
122 egBSTreeL = mkNodeBT('D',
123                 mkNodeBT('B',
124                   mkNodeBT('A',mkEmptyBT(),mkEmptyBT()),
125                   mkNodeBT('C',mkEmptyBT(),mkEmptyBT())
126                 ),
127                 mkNodeBT('F',
128                   mkNodeBT('E',mkEmptyBT(),mkEmptyBT()),
129                   mkNodeBT('G',mkEmptyBT(),mkEmptyBT())
130                 )
131               )
```

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Answer 2 Python Representation — egBSTreeLL

```
146 egBSTreeLL = mkNodeBT('B',
147                   mkNodeBT('A',mkEmptyBT(),mkEmptyBT()),
148                   mkNodeBT('C',mkEmptyBT(),mkEmptyBT())
149                 )
```

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Binary Trees

Tree Traversals

- Many applications require visiting each node in a binary tree and doing some processing.
- This could be adding quantities to find a total, identifying the number of nodes with a particular property and so on.
- There are several common patterns of visiting each node or traversing a tree
  - Depth first where the search tree is deepened as much as possible on each child before visiting the next sibling
  - Breadth first where we visit every node on a level before visiting the next level
- Each traversal takes a tree and returns a list of items at the nodes of the tree

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Tree Traversals

Depth First

- **In-Order traversal of tree `t`**
  1. If `t` is an empty tree then return the empty list
  2. Otherwise do an In Order traversal of the left subtree of `t` then append a list just containing the data item at the root of `t` followed by an In Order traversal of the right subtree of `t`
- **Pre-Order traversal of tree `t`**
  - As In-Order but output a list with the item at the root of `t` before traversing the two subtrees
- **Post-Order traversal of tree `t`**
  - As Pre-Order but output a list with the item at the root of `t` after traversing the two subtrees
- *In-Order*, *Pre-Order* and *Post-Order* traversals are collectively termed *Depth First Traversals*
- We first provide the usual recursive implementations — in a later section we translate the recursive versions into iterative versions

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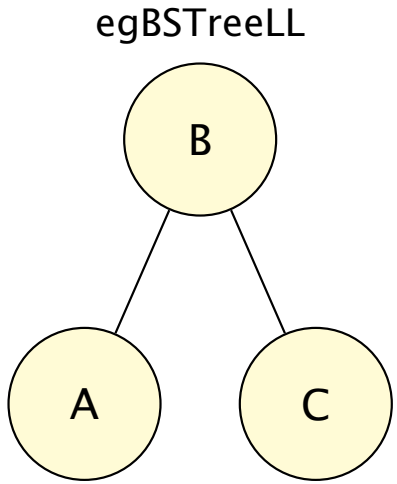
Depth First Traversal

Example

- Tree egBSTreeLL Python code at line 146 on slide 60



- The *Depth first* traversals are implemented in Python by inOrderBT(), preOrderBT() and postOrderBT()

```
Python3>>> inOrderBT(egBSTreeLL)
['A', 'B', 'C']
Python3>>> preOrderBT(egBSTreeLL)
['B', 'A', 'C']
Python3>>> postOrderBT(egBSTreeLL)
['A', 'C', 'B']
```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Depth First Traversals

Python

```
311  def inOrderBT(t) :
312    if isEmptyBT(t) :
313      return []
314    else :
315      return (inOrderBT(getLeftBT(t)) + [getDataBT(t)]
316              + inOrderBT(getRightBT(t)))

318  def preOrderBT(t) :
319    if isEmptyBT(t) :
320      return []
321    else :
322      return ([getDataBT(t)] + preOrderBT(getLeftBT(t))
323              + preOrderBT(getRightBT(t)))

325  def postOrderBT(t) :
326    if isEmptyBT(t) :
327      return []
328    else :
329      return (postOrderBT(getLeftBT(t))
330              + postOrderBT(getRightBT(t)) + [getDataBT(t)])
```


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Activity 3 Depth First Traversals

- Give the lists of items in an in-order traversal of egBSTree, egBSTree1, egBSTree2, egBSTree3
- Give the lists of items in a pre-order traversal of egBSTree, egBSTree1, egBSTree2, egBSTree3
- Give the lists of items in a post-order traversal of egBSTree, egBSTree1, egBSTree2, egBSTree3
- Depth first traversal code is from line 311 on slide 64 (Python)
- Binary tree code is from line 70 on slide 64 (Python)

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Answer 3 Depth First Traversals — In-Order

```
Python3>>> inOrderBT(egBSTree)
['A', 'B', 'C', 'D', 'E', 'F', 'G',
 'H', 'I', 'J', 'K', 'L', 'M', 'N', 'O']
Python3>>> inOrderBT(egBSTree1)
['D', 'H', 'J', 'L', 'M', 'N', 'O']
Python3>>> inOrderBT(egBSTree2)
['A', 'B', 'C', 'D', 'E', 'F', 'G',
 'H', 'I', 'J', 'L', 'N']
Python3>>> inOrderBT(egBSTree3)
['A', 'B', 'C', 'D', 'F', 'G', 'H',
 'L', 'M', 'N', 'O']
```

- (Line breaks introduced for layout)
- Answer 3 continued on next slide

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Answer 3 Depth First Traversals — Pre-Order

```
Python3>>> preOrderBT(egBSTree)
['H', 'D', 'B', 'A', 'C', 'F', 'E',
 'G', 'L', 'J', 'I', 'K', 'N', 'M', 'O']
Python3>>> preOrderBT(egBSTree1)
['H', 'D', 'L', 'J', 'N', 'M', 'O']
Python3>>> preOrderBT(egBSTree2)
['H', 'D', 'B', 'A', 'C', 'F', 'E',
 'G', 'L', 'J', 'I', 'N']
Python3>>> preOrderBT(egBSTree3)
['H', 'D', 'B', 'A', 'C', 'F', 'G',
 'L', 'N', 'M', 'O']
```

▶ Answer 3 continued on next slide

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Answer 3 Depth First Traversals — Post-Order

```
Python3>>> postOrderBT(egBSTree)
['A', 'C', 'B', 'E', 'G', 'F', 'D',
 'I', 'K', 'J', 'M', 'O', 'N', 'L', 'H']
Python3>>> postOrderBT(egBSTree1)
['D', 'J', 'M', 'O', 'N', 'L', 'H']
Python3>>> postOrderBT(egBSTree2)
['A', 'C', 'B', 'E', 'G', 'F', 'D',
 'I', 'J', 'N', 'L', 'H']
Python3>>> postOrderBT(egBSTree3)
['A', 'C', 'B', 'G', 'F', 'D', 'M',
 'O', 'N', 'L', 'H']
```

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Tree Traversals

Breadth First

- The M269 book section 16.3.5 gives an iterative version of a breadth first traversal but only mentions a recursive version briefly
- We shall start with a recursive version and transform that by stages into the iterative version in the book
- I find it easier to think of the recursive version first — you should observe how people think they think about programming
- First we do some exercises

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Binary Tree Exercises

Activity 4 Breadth First Traversals

- A *level order* traversal of a binary tree takes a tree and returns the list of levels
- Each level is the list of items at that level
- Give the list of levels for:
- `egBSTree`
- `egBSTreeL`
- `egBSTree1`
- `egBSTree2`
- `egBSTree3`

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Answer 4 Breadth First Traversals

- Answer 4 Breadth First Traversals

```
Python3>>> levelOrderBT(egBSTree)
[['H'], ['D', 'L'], ['B', 'F', 'J', 'N'],
  ['A', 'C', 'E', 'G', 'I', 'K', 'M', 'O']]
Python3>>> levelOrderBT(egBSTreeL)
[['D'], ['B', 'F'], ['A', 'C', 'E', 'G']]
Python3>>> levelOrderBT(egBSTree1)
[['H'], ['D', 'L'], ['J', 'N'], ['M', 'O']]
Python3>>> levelOrderBT(egBSTree2)
[['H'], ['D', 'L'], ['B', 'F', 'J', 'N'],
  ['A', 'C', 'E', 'G', 'I']]
Python3>>> levelOrderBT(egBSTree3)
[['H'], ['D', 'L'], ['B', 'F', 'N'], ['A', 'C', 'G', 'M', 'O']]
Python3>>>
```

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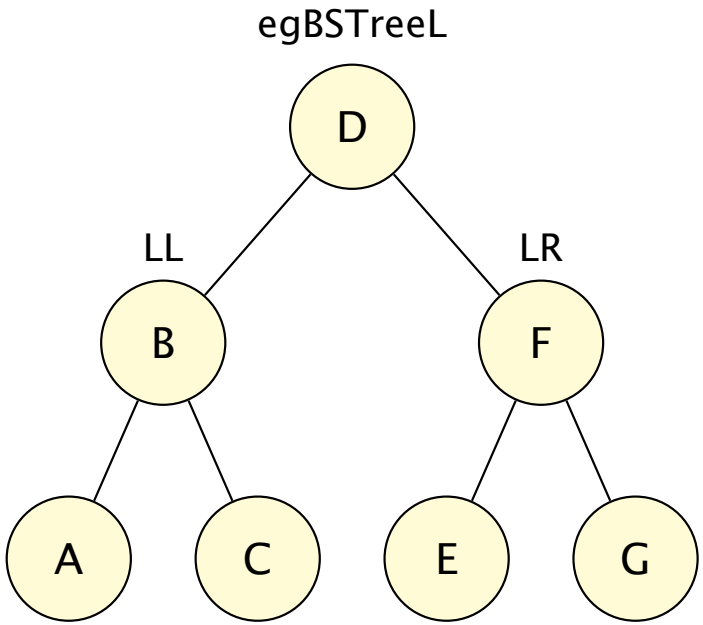
Answer 4 Breadth First Traversals (c)

- Answer 4 Breadth First Traversals

```
Python3>>> levelOrderBT(egBSTreeL)
[['D'], ['B', 'F'], ['A', 'C', 'E', 'G']]
```



- Answer 4 continued on next slide

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Breadth First Traversal

Version 01 (a)

- The first version will be recursive and driven by the structure of trees and will start by writing `levelOrder()` which takes a binary tree and returns a list of levels — a `level` is a list of items at the level

(1) An empty tree has an empty list of levels (level zero)

(2) A non-empty tree has the list of the root item followed by combining the two lists of the levels for the two sub-trees

- We will call the function that combines the two lists of levels `longZipMerge()` since it is similar to the Python library `zip()` function

```
428  def levelOrderBT(t : Tree) -> [[T]] :
429    if isEmptyBT(t) :
430      return []
431    else :
432      x = getDataBT(t)
433      left = getLeftBT(t)
434      right = getRightBT(t)
435      return ([[x]] +
436        longZipMerge(levelOrderBT(left),levelOrderBT(right)))
```


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Version 01 (b)

- `longZipMerge()` is a variant on the Python library function `zip()`
- `zip()` iterates over several iterables in parallel, producing tuples with an item from each one.
- `longZipMerge()` takes two lists and returns a new list with merged pairs of items from each list which is a level order traverse of the subtrees
- The two lists do not need to be of the same length — any excess is just appended to the merged result so far

```
438  def longZipMerge(xss : [[T]],yss : [[T]]) -> [[T]] :
439    if xss == [] :
440      return yss
441    elif yss == [] :
442      return xss
443    else :
444      return ([xss[0] + yss[0]] + longZipMerge(xss[1:],yss[1:]))
```


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Version 01 (c)

- Evaluation of `levelOrderBT(egBSTreeL)`

```
levelOrderBT(egBSTreeLL)
= [['B']]                                  # by line 431
   + longZipMerge(levelOrderBT(egBSTreeLLL),
                  levelOrderBT(egBSTreeLLR))
= [['B']]                                  # by lines 431,439
   + longZipMerge([['A']],[['C']])
= [['B']] + [['A','C']]                    # by line 441
= [['B'],['A','C']]
```

```
levelOrderBT(egBSTreeLR)
= [['F'],['E','G']]                        # as above
```

```
levelOrderBT(egBSTreeL)
= [['D']]                                  # by line 431
   + longZipMerge(levelOrderBT(egBSTreeLL),
                  levelOrderBT(egBSTreeLR))
= [['D']]                                  # as above
   + longZipMerge([['B'],['A','C']],
                  [['F'],['E','G']])
= [['D']]                                  # by line 441
   + [['B','F'],['A','C','E','G']]
= [['D'],['B','F'],['A','C','E','G']]  # correct - check the steps
```


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Version 01 (d)

- Testing can only show the presence of bugs but not the absence of bugs (Edsger W Dijkstra Quotes)
- We shall now investigate a similar program with a subtle error

```
465  def levelOrderBT01(t : Tree) -> [[T]] :
466    if isEmptyBT(t) :
467      return []
468    else :
469      x = getDataBT(t)
470      left = getLeftBT(t)
471      right = getRightBT(t)
472      return ([x] +
473        longZipMerge01(levelOrderBT01(left),levelOrderBT01(right)))

475  def longZipMerge01(xss : [[T]],yss : [[T]]) -> [[T]] :
476    if xss == [] :
477      return yss
478    elif yss == [] :
479      return xss
480    else :
481      return ([xss[0],yss[0]] + longZipMerge01(xss[1:],yss[1:]))
```


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Version 01 (e)

- We first do a few tests

```
Python3>>> levelOrderBT(egBSTreeLL)
[['B'], ['A', 'C']]
Python3>>> levelOrderBT01(egBSTreeLL)
['B', 'A', 'C']
Python3>>> levelOrderBT(egBSTree1)
[['H'], ['D', 'L'], ['J', 'N'], ['M', 'O']]
Python3>>> levelOrderBT01(egBSTree1)
['H', 'D', 'L', 'J', 'N', 'M', 'O']
```

- Correct order but a list of items not a list of levels

```
Python3>>> levelOrderBT(egBSTreeL)
[['D'], ['B', 'F'], ['A', 'C', 'E', 'G']]
Python3>>> levelOrderBT01(egBSTreeL)
['D', 'B', 'F', 'A', 'E', 'C', 'G']
```

- Wrong order — we now do an evaluation to see where the error is

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Version 01 (f)

- Evaluation of `levelOrderBT01(egBSTreeL)`

```
levelOrderBT01(egBSTreeLL)
= ['B']                                    # by line 468
   + longZipMerge01(levelOrderBT01(egBSTreeLLL),
                    levelOrderBT01(egBSTreeLLR))
= ['B']                                    # by lines 468,476
   + longZipMerge01(['A'],['C'])
= ['B'] + ['A','C']                        # by line 478
= ['B','A','C']
```

```
levelOrderBT01(egBSTreeLR)
= ['F','E','G']                            # as above
```

```
levelOrderBT01(egBSTreeL)
= ['D']                                    # by line 468
   + longZipMerge01(levelOrderBT01(egBSTreeLL),
                    levelOrderBT01(egBSTreeLR)))
= ['D']                                    # as above
   + longZipMerge01(['B','A','C'],
                    ['F','E','G'])
= ['D']                                    # by line 478
   + ['B','F','A','E','C','G']
= ['D','B','F','A','E','C','G']            # notice the error ?
```


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Version 01 (g)

- `levelOrderBT01()` is not only of the wrong type but produces the wrong order except for a limited number of trees
- The Python type annotations are only checked for syntax but not for correctness
- We get the final `breadthBT01()` by flattening the list of levels
- This uses a *list comprehension* as a shorthand for nested loops — see explanation below

```
454  def flattenLevels(levels : [[T]]) -> [T] :
455    return ([elem for level in levels for elem in level])

457  def breadthBT01(t : Tree) -> [T] :
458    return flattenLevels(levelOrderBT(t))
```


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Version 01 (h) List Comprehensions

- Python List comprehensions (tutorial), List comprehensions (reference) — a neat way of expressing iterations over a list
- Example (a) Square the even numbers between 0 and 9
- Example (b) Generate a list of pairs which satisfy some condition

```
Python3>>> [x ** 2 for x in range(10) if x % 2 == 0]
[0, 4, 16, 36, 64]
Python3>>> [(x,y) for x in range(4)
...               for y in range(4)
...               if x % 2 == 0
...                  and y % 3 == 0]
[(0, 0), (0, 3), (2, 0), (2, 3)]
Python3>>>
```

- In general

```
 [expr for target1 in iterable1 if cond1
       for target2 in iterable2 if cond2 ...
       for targetN in iterableN if condN ]
```


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Version 01 (i) List Comprehensions

- Instead of the list comprehension, `flattenLevels()` could be defined with an accumulating list and nested loops.
- M269 does not mention list comprehensions so you would have to decide whether they are worth mentioning

```
490 def flattenLevelsA(levels : [[T]]) -> [T] :
491   accumList = []
492   for level in levels :
493     for elem in level :
494       accumList = accumList + [elem]
495   return accumList
```


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Version 02 (a)

- We now have a correct program `breadthBT01()` but this does lots of (how many?) traversals of the data — we may want a more efficient version and hence we transform our program
- Version 02 uses a helper function `bfTraverse(vs,ts)` which takes a list of item seen, `vs`, and a list (or queue) of trees to be visited, `ts`
- As we visit a node, we add its subtree to the queue, `ts`

```
501 def breadthBT02(t : Tree) -> [T] :
502   return bfTraverse([],[t])

504 def bfTraverse(vs : [T],ts : [Tree]) -> [T] :
505   if ts == []:
506     return vs
507   elif isEmptyBT(ts[0]):
508     return bfTraverse(vs,ts[1:])
509   else:
510     return (bfTraverse(vs + [getDataBT(ts[0])],
511                 ts[1:] + [getLeftBT(ts[0]),getRightBT(ts[0])]))
```


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Version 03 (a)

- Version 02 removed some of the recursion by enqueueing trees to be visited
- This version has the disadvantage that no output until all the nodes are visited, which could mean a long wait or never if the tree is infinite
- Version 03 enables a lazier approach — Python could use a Generator expression or augment the code with Yield expressions (both not used in M269) but other languages, such as Haskell use lazy evaluation by default

```
513 def breadthBT03(t : Tree) -> [T] :
514   return lbfBT([t])

516 def lbfBT(ts : [Tree]) -> [T] :
517   if ts == []:
518     return []
519   elif isEmptyBT(ts[0]):
520     return lbfBT(ts[1:])
521   else:
522     return ([getDataBT(ts[0])]
523         + lbfBT(ts[1:] + [getLeftBT(ts[0]),getRightBT(ts[0])]))
```


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Version 04 (a)

- Version 03 has the only recursive call as (almost) the last thing
- So we can implement this with a `while` loop

```
527 def breadthBT04(t : Tree) -> [T] :
528   ts = [t] # Trees to visit
529   vs = []  # Values seen
530   while (ts != []) :
531     if not (isEmptyBT(ts[0])) :
532       vs = vs + [getDataBT(ts[0])]
533       ts = ts[1:] + [getLeftBT(ts[0]),getRightBT(ts[0])]
534     else :
535       ts = ts[1:]
536   return vs
```


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Activity 5 Breadth First Error

- There is an error in the following version — what is the error ?
- Why would the print statements not help ?
- Why don't the Python type annotations help ?

```
543 def breadthBT04A(t : Tree) -> [T] :
544   ts = [t] # Trees to visit
545   vs = []  # Values seen
546   while not (isEmptyBT(ts)) :
547     print('ts = ',ts)
548     if not (isEmptyBT(ts[0])) :
549       print('len(ts) = ',len(ts))
550       print('getDataBT(ts[0]) = ',getDataBT(ts[0]))
551       vs = vs + [getDataBT(ts[0])]
552       ts = ts[1:] + [getLeftBT(ts[0]),getRightBT(ts[0])]
553     else :
554       print('ts[0] is empty','len(ts) = ',len(ts))
555       ts = ts[1:]
556   return vs
```

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Answer 5 Breadth First Error

- The error is the `while` condition at line `546`
- `isEmptyBT(ts)` will never return `True` since `ts` is a list of trees
- The article version of these notes contains an output of the print statements and the error report
- The error is reported at line `548` as *IndexError: list index out of range*
- So the print statements do not show the real error
- The Python interpreter does not check the type annotations for correctness, just the syntax
- Remember that Python is a weakly typed language

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Version 05 (a)

- There is one disadvantage of version 01
- The program traverses the entire left subtree before traversing the right subtree
- Bad news for large trees and very bad for infinite trees
- This version produces the traversal level by level

```
563 def labelsAtDepth(d : int, t : Tree) -> [T] :
564   if isEmptyBT(t) :
565     return []
566   else :
567     x = getDataBT(t)
568     left = getLeftBT(t)
569     right = getRightBT(t)
570     if d == 0 :
571       return [x]
572     else :
573       return (labelsAtDepth((d-1),left) + labelsAtDepth((d-1),right))
```


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Breadth First Traversal

Version 05 (b)

- Breadth first traversal with `labelsAtDepth`
- Version based on Sannella et al (2022, page 261) *Introduction to Computation: Haskell, Logic and Automata*

```
577 def bfTraverseByLevels(t : Tree) -> [T] :
578   return bfTbyL(0,t)

580 def bfTbyL(d : int, t : Tree) -> [T] :
581   xs = labelsAtDepth(d,t)
582   if xs == [] :
583     return []
584   else :
585     return (xs + bfTbyL((d+1),t))
```


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Iterative Tree Traversals

Recursion Removal

- We have used recursion in our implementation of algorithms on binary trees
- This has made it easier to produce correct and fairly simple implementations
- This is mainly because the binary tree data structure is itself defined recursively
- A binary tree is either an empty tree or a node with a data item and two subtrees.
- However the efficiency of this approach will depend on how the chosen programming language is implemented.
- We are using Python and, while Python permits recursion, it does not do some of the optimisations available in other languages, especially pure functional languages (such as Haskell).
- Hence you may find some Python texts down play the use of recursion.

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Recursion Removal

- It is always possible to convert a recursive program into one that just uses iteration with `while` loops or (possibly) `for` loops
- We give below examples of the depth first tree traversals translated from their recursive forms to non-recursive.
- Note that this subsection is for illustration only and you would not be expected to be able to reproduce the code or convert other recursive code.

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Iterative Tree Traversals

inOrder Traversal (1)

- Here is the original recursive version (from line 311 on slide 64)

```
311 def inOrderBT(t) :
312   if isEmptyBT(t) :
313     return []
314   else :
315     return (inOrderBT(getLeftBT(t)) + [getDataBT(t)]
316             + inOrderBT(getRightBT(t)))
```


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Iterative Tree Traversals

inOrder Traversal (2)

- We start with the recursive version but with an accumulating result.

```
335 def inOrderBTO(t) :
336   result = []
337   if not isEmptyBT(t) :
338     result = result + (inOrderBTO(getLeftBT(t)))
339     result.append(getDataBT(t))
340     result = result + (inOrderBTO(getRightBT(t)))
341   return result
```


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Iterative Tree Traversals

inOrder Traversal (3)

- Turn the final (*almost* tail recursive) call into a `while` loop

```
343 def inOrderIterBT1(t) :
344   result = []
345   while not isEmptyBT(t) :
346     result = result + (inOrderIterBT1(getLeftBT(t)))
347     result.append(getDataBT(t))
348     t = getRightBT(t)
349   return result
```

- The term *almost* since the last operation is the addition `(+)` but that could be wrapped into the last call

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Iterative Tree Traversals

inOrder Traversal (4)

- There is now one recursive call.
- Create a stack to store the function call context.
- In the loop have a conditional to determine whether to store a new context and make the left sub tree the current node or if we are returning, with the appropriate code.

```
351 def inOrderIterBT2(t) :
352   result = []
353   stack = []
354   while (not (stack == []) or not isEmptyBT(t)) :
355     if not isEmptyBT(t) :
356       stack.append(t)
357       t = getLeftBT(t)
358     else:
359       t = stack.pop()
360       result.append(getDataBT(t))
361       t = getRightBT(t)
362   return result
```


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inOrder Traversal (5)

Algorithm Description

- `inOrderIterBT2` takes a tree `t` and returns a list of items at the nodes, depth first from left to right
  1. Initialise `result` to an empty list, and `stack`, for the stack of trees to visit, to an empty list
  2. While `stack` is not empty or `t` is not the empty tree
     - 2.1 If `t` is not empty, append `t` to `stack` and assign `t` to be its left sub tree — this is the left recursion
     - 2.2 Otherwise make `t` the top of the `stack` (and remove it), append the item at its node to `result` and make `t` to be its right sub tree
  3. Finally return `result`

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Iterative Tree Traversals

preOrder Traversal (1)

- Here is the original recursive version (from line 318 on slide 64)

```
318 def preOrderBT(t) :
319   if isEmptyBT(t) :
320     return []
321   else :
322     return ([getDataBT(t)] + preOrderBT(getLeftBT(t))
323             + preOrderBT(getRightBT(t)))
```


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Iterative Tree Traversals

preOrder Traversal (2)

- Start with recursive version with accumulating result.

```
364 def preOrderBTO(t) :
365   result = []
366   if not isEmptyBT(t) :
367     result.append(getDataBT(t))
368     result = result + (preOrderBTO(getLeftBT(t)))
369     result = result + (preOrderBTO(getRightBT(t)))
370   return result
```


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preOrder Traversal (3)

- Turn the final (*almost* tail recursive) call into a `while` loop.

```
372 def preOrderIterBT1(t) :
373   result = []
374   while not isEmptyBT(t) :
375     result.append(getDataBT(t))
376     result = result + (preOrderIterBT1(getLeftBT(t)))
377     t = getRightBT(t)
378   return result
```


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preOrder Traversal (4)

- There is now one recursive call.
- Create a stack to store the function call context.
- In the loop have a conditional to determine whether to store a new context and make the left sub tree the current node or if we are returning, with the appropriate code

```
380 def preOrderIterBT2(t) :
381   result = []
382   stack = []
383   while (not (stack == []) or not isEmptyBT(t)) :
384     if not isEmptyBT(t) :
385       result.append(getDataBT(t))
386       stack.append(t)
387       t = getLeftBT(t)
388     else :
389       t = stack.pop()
390       t = getRightBT(t)
391   return result
```


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Iterative Tree Traversals

postOrder Traversal (1)

- Here is the original recursive version (from line 325 on slide 64)

```
325 def postOrderBT(t):
326   if isEmptyBT(t):
327     return []
328   else:
329     return (postOrderBT(getLeftBT(t))
330             + postOrderBT(getRightBT(t)) + [getDataBT(t)])
```


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Iterative Tree Traversals

postOrder Traversal (2)

- Start with recursive version with accumulating result.

```
393 def postOrderBT0(t) :
394   result = []
395   if not isEmptyBT(t) :
396     result = result + (postOrderIterBT1(getLeftBT(t)))
397     result = result + (postOrderIterBT1(getRightBT(t)))
398     result.append(getDataBT(t))
399   return result
```


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postOrder Traversal (3)

- There is now no final (tail recursive) call.

```
401 def postOrderIterBT1(t) :
402   result = []
403   if isEmptyBT(t) :
404     return result
405   stack = []
406   while not (stack == []) or (not isEmptyBT(t)) :
407     while not isEmptyBT(t) :
408       if not isEmptyBT(getRightBT(t)) :
409         stack.append(getRightBT(t))
410       stack.append(t)
411       t = getLeftBT(t)
412     t = stack.pop()
413     if ((not isEmptyBT(getRightBT(t)))
414         and (stack != [] and stack[-1] is getRightBT(t))) :
415       tr = stack.pop()
416       stack.append(t)
417       t = getRightBT(t)
418     else :
419       result.append(getDataBT(t))
420       t = mkEmptyBT()  # To avoid infinite loop - it is t.rightBT
421   return result
```


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postOrder Traversal (4) — Algorithm Description

1. Initialise `result` to an empty list.
2. If `t` is empty then return `result` (not really needed since the loop would take care of this)
3. Initialise `stack`, for the stack of trees to visit, to an empty list
4. While `stack` is not empty or `t` is not the empty tree
   - 4.1 While `t` is not the empty tree
     - If the right sub tree of `t` is not empty, push it on to `stack`
     - Append `t` to `stack`
     - Assign `t` its left sub tree
   - 4.2 Pop a node from `stack` and set it as `t`
   - 4.3 If the popped node has a non empty right child and the right child is at the top of `stack`
     - Remove the right child from the `stack`
     - Push the current node `t` on to `stack`
     - Set `t` to be `t`'s right child
   - 4.4 Otherwise
     - Append the data at the root of `t` to `result`
     - Set `t` to `Empty()` — marking `t` as visited, prevents infinite looping (it is `t.rightBT`)
5. Finally return `result`

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Concluding Points

- Recursive versions are easier to get right.
- Iterative versions mimic the stack of recursive function calls.
- Other non-recursive versions use different data structures with pointers to parent nodes. The code is still more complex (and error prone) compared to the recursive versions.

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Binary Search Trees

3 Binary Search Trees

- Binary trees with the *binary search tree* property
- Inserting a node
- Other BST operations
- Deletion — investigating choices

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Binary Search Trees

Definition

- A binary search tree (BST) is a binary tree with the *binary search tree* property:
  1. The left sub tree contains nodes with keys less than the root node
  2. The right sub tree contains nodes with keys greater than the root node
  3. The left and right sub trees must also be binary search trees
  4. No nodes with duplicate keys
  5. An empty tree is a binary search tree
  6. Nothing else is a binary search tree
- The data at each node is to contain a key and any values. The operations required for a BST will include:

  `insertBST()`, `inBST()`, `isBSTree()`, `insertListBST()`, `buildBST()`, `deleteBST()`

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Binary Search Trees

Motivation

- A *perfect* binary tree of height h will have $2^h - 1$ nodes
- This means that there will be at most $\log_2(n+1)$ steps from the root of the tree to any node in the tree.
- This provides the basis for efficient searching if we give an appropriate structure to the tree — a *Binary Search Tree (BST)*
- However we have to keep the BST as near to a perfect tree otherwise we lose the advantage

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Perfect Trees

Activity 6 Nodes of Perfect Tree

- Justify the statement that a *perfect* binary tree of height h will have $2^h - 1$ nodes

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Perfect Trees

Answer 6 Nodes of Perfect Tree

- There are many ways of showing that a *perfect* binary tree of height h will have $2^h - 1$ nodes — here is one way
- Let N_h be the number of nodes in a perfect tree and L_h be the number of leaves in the same tree.
- Then we have $L_0 = 0$, $L_1 = 1$, $L_2 = 2$, $L_3 = 4, \ldots$ and in general $L_h = 2^{h-1}$, $h \geqslant 1$
- Now $N_h = L_1 + L_2 + \cdots + L_h = 2^0 + 2^1 + \cdots + 2^{h-1}$
- $2N_h = 2^1 + 2^2 + \cdots + 2^{h-1} + 2^h$
- Subtract the N_h from $2N_h$ and we get $N_h = 2^h - 1$
- Notice that $N_h = 1 + 2 \times N_{h-1}$ — when we consider the performance we will use similar recurrence relations

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Inserting a Node

Description

- The function that takes a item (key and payload) and an existing BST has to return a new binary tree with the item inserted and the new tree must be a BST. We deal with each possible binary tree: an empty tree and a non-empty tree:
- To insert an item into an empty tree, we return a new tree which is a singleton node with the item and two empty sub-trees.
- To insert an item, with key k, into a tree which is a node with an item with key p and two sub-trees then we have two possibilities
  - If k is less than p then insert the item in the left sub-tree
  - If k is greater than p then insert the item in the right sub-tree
- We are going to assume duplicate keys are not allowed

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Binary Search Tree

Inserting a Node

```
564 def insertBST(x,t):
565   if isEmptyBT(t):
566     return mkNodeBT(x,mkEmptyBT(),mkEmptyBT())
567   else:
568     y = getDataBT(t)
569     if x < y:
570       return mkNodeBT(y,insertBST(x,getLeftBT(t)),getRightBT(t))
571     elif x > y:
572       return mkNodeBT(y,getLeftBT(t),insertBST(x,getRightBT(t)))
573     else:
574       return t
```


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Binary Search Tree

Activity 7 Inserting an Item

- Draw diagrams of the binary search trees that result from inserting an item with key 28 into each of the three BSTs in the diagrams below of `insBSTreeA`, `insBSTreeB`, `insBSTreeC`

insBSTreeA

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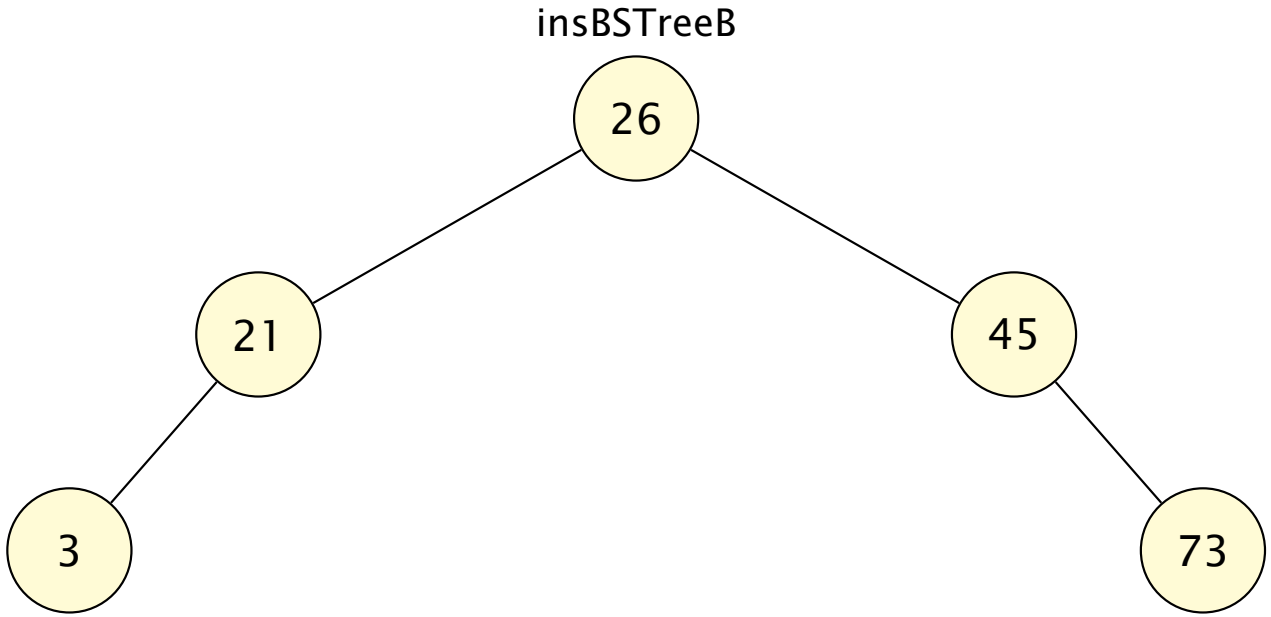
Binary Search Tree

Activity 7 Inserting an Item — insBSTreeB

insBSTreeB



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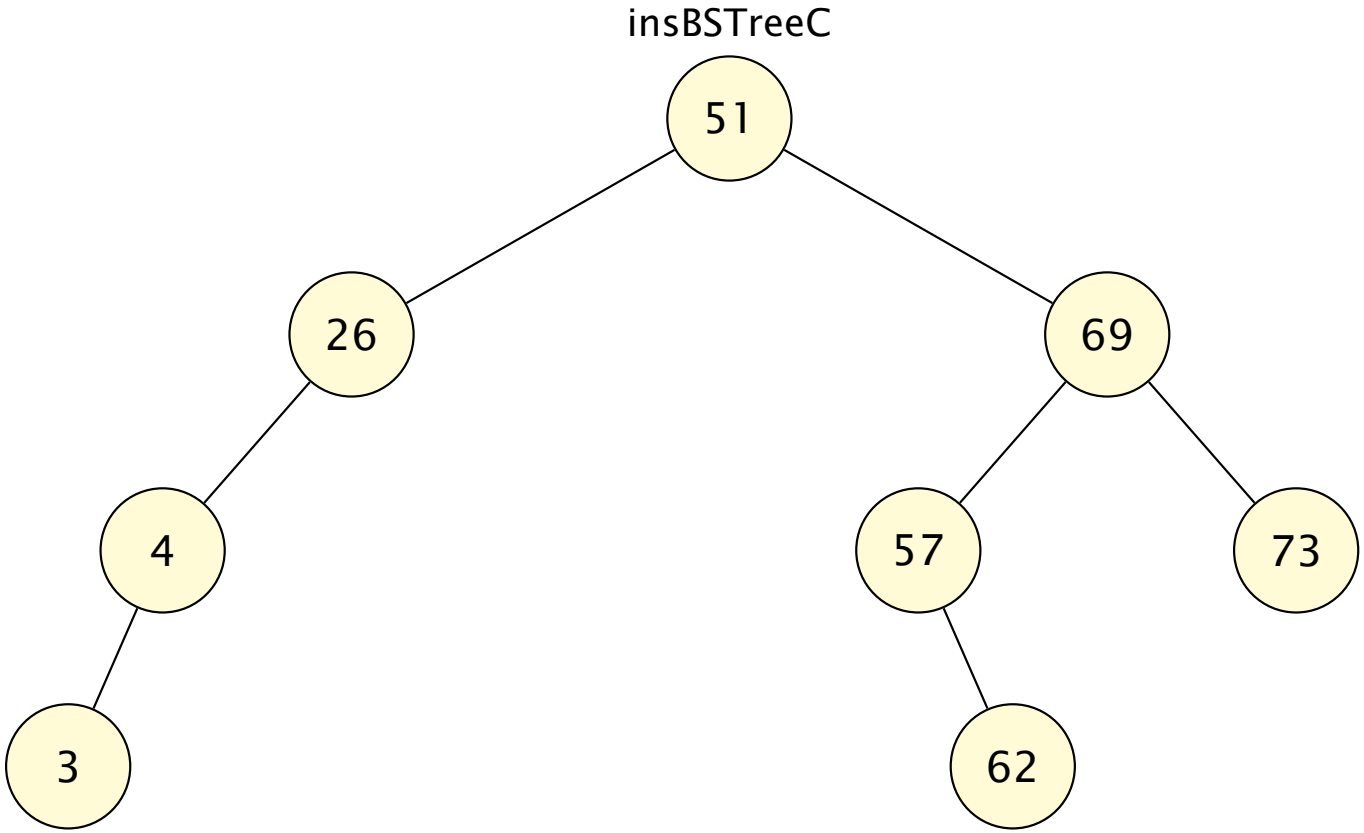
Binary Search Tree

Activity 7 Inserting an Item — insBSTreeC

insBSTreeC



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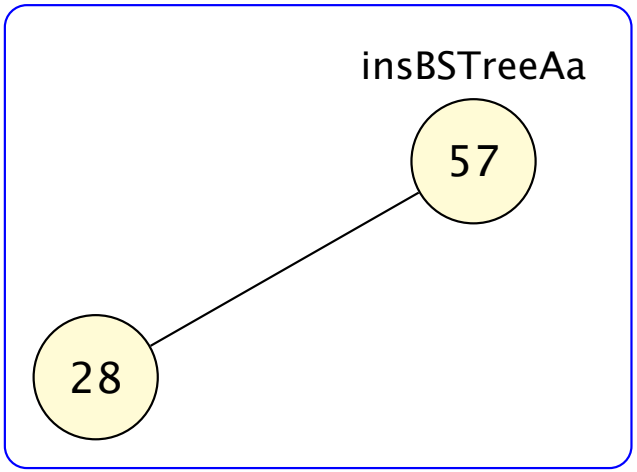
Binary Search Tree

Answer 7 Inserting an Item — insBSTreeA

insBSTreeAa



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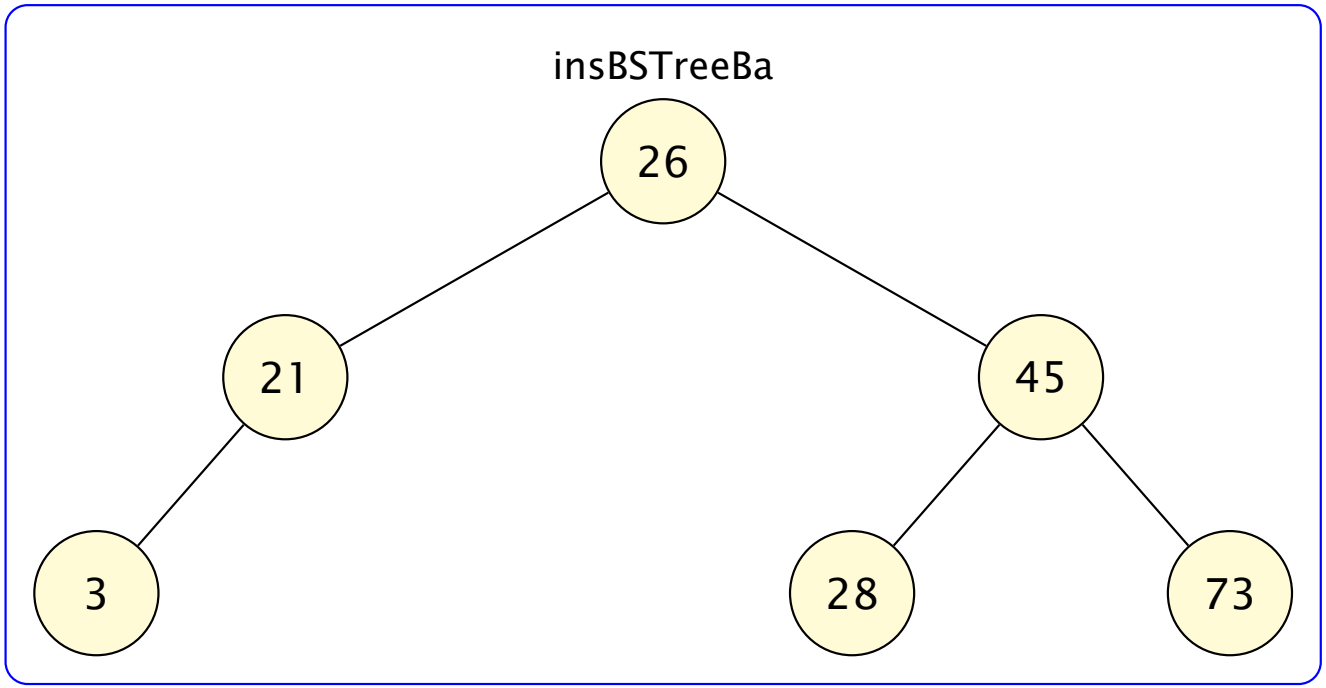
Binary Search Tree

Answer 7 Inserting an Item — insBSTreeB

insBSTreeBa



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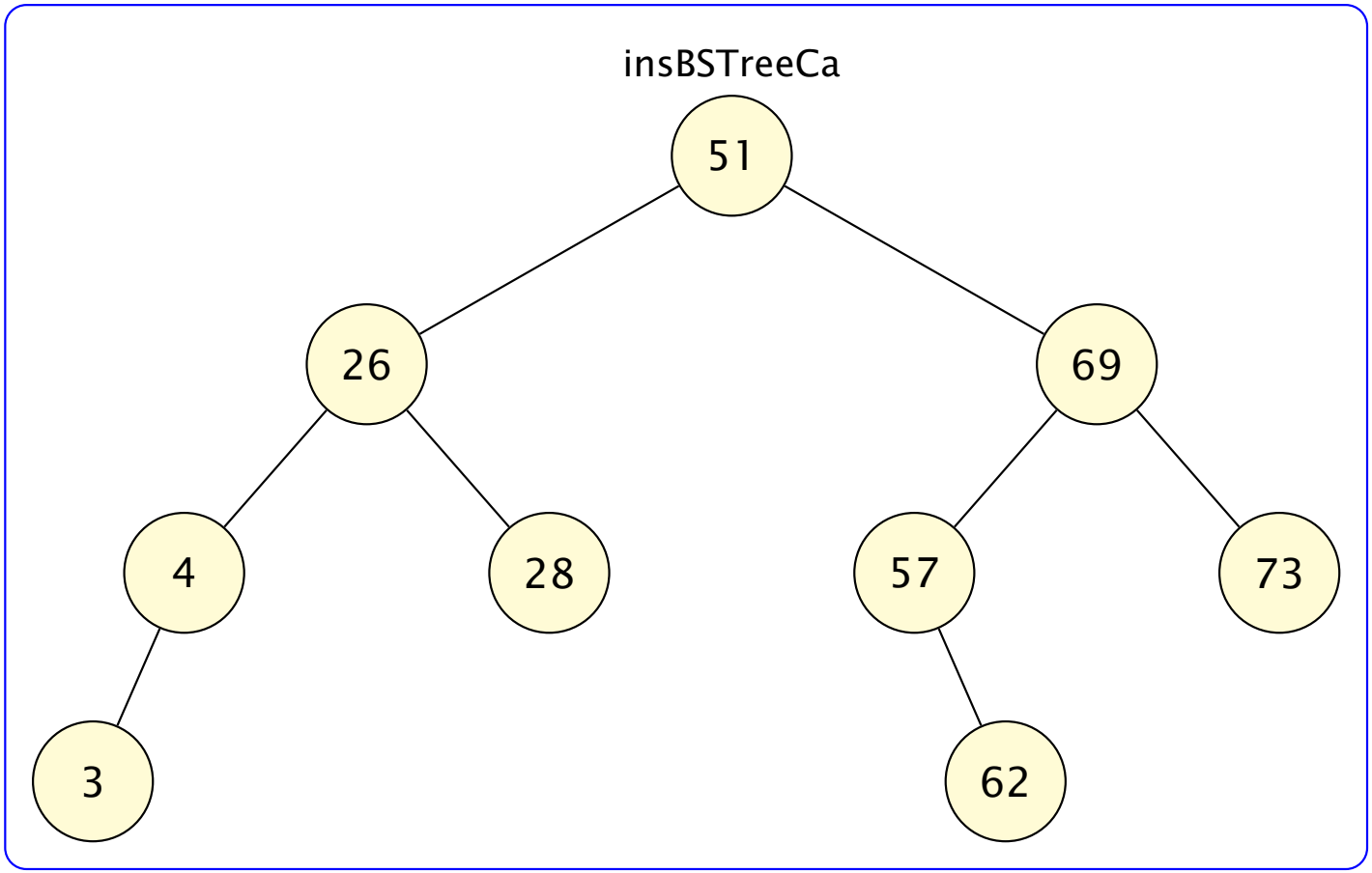
Answer 7 Inserting an Item — insBSTreeC

insBSTreeCa



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Binary Search Tree

Activity 8 Membership

- Write Python code for a function, `inBST(k,t)` which take a key, *k*, and a BST, *t* and returns `True` if an item with key *k* is in the tree and `False` otherwise

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Binary Search Tree

Answer 8 Membership

```
576 def inBST(k,t):
577   if isEmptyBT():
578     return False
579   else:
580     p = getDataBT(t)
581     if k < p:
582       return inBST(k,getLeftBT(t))
583     elif k > p:
584       return inBST(k,getRightBT(t))
585     else:
586       return True
```

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Binary Search Tree Operations

Testing if a Binary Tree is a BST

- One strategy for this might be to do an in-order traversal of the tree and check that the list returned was an ordered list.
- The ordering relation is (<) for strict ordering and no duplicates

```
588 def isBSTree(t):
589   return orderedList(inOrderBT(t))

591 def orderedList(xs):
592   return (len(xs) <= 1
593           or (xs[0] < xs[1] and orderedList(xs[1:])))
```


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Binary Search Tree Operations

Building a Binary Search Tree from a list of items

- We *could* insert a list of items one by one from the list in turn — here is the Python code:

```
595 def insertListBST(xs,t) :
596   if xs == [] :
597     return t
598   else :
599     return insertListBST (xs[1:], insertBST(xs[0],t))
```

- However, see what happens when we insert various lists — how *compact* is the resulting tree ?

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Binary Search Trees

Activity 9 Insert List

- For the following lists of keys, draw the diagrams of the trees produced when `insertListBST()` is used to produce the trees from the lists inserting the keys into an initially empty tree
- `keys1 = [10, 4, 32, 12, 9, 55, 92, 97, 36, 41, 34]`
- `keys2 = [4, 9, 10, 12, 32, 97, 92, 55, 41, 34, 32]`
- `keys3 = [34, 10, 9, 4, 32, 12, 55, 41, 36, 97, 92]`

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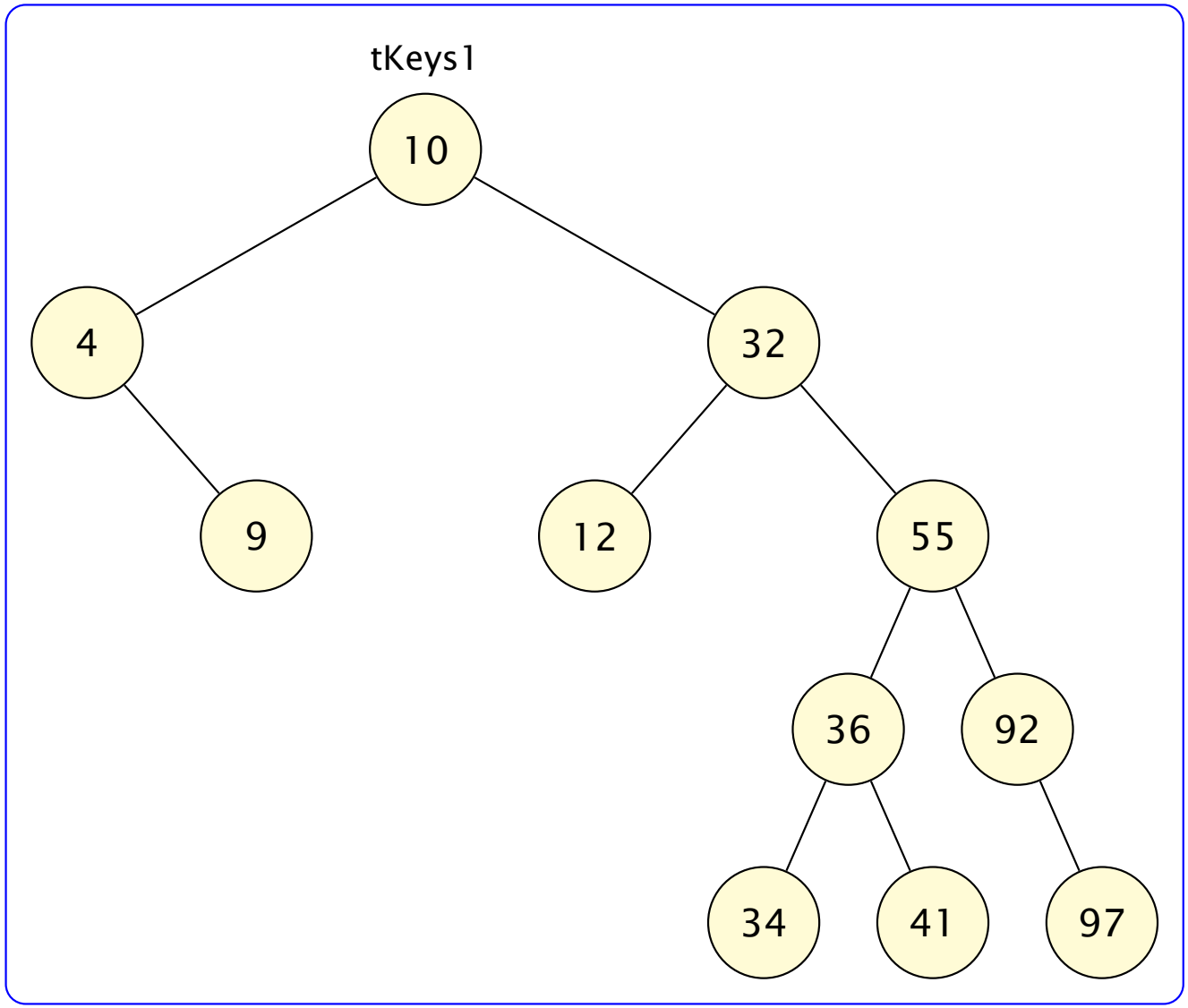
Binary Search Trees

Answer 9 Insert List `tKeys1 = insertListBST(keys1, mkEmptyBT())`



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Binary Search Trees

Answer 9 Insert List `tKeys1 = insertListBST(keys1, mkEmptyBT())`

```
tKeys1 = mkNodeBT(10,
           mkNodeBT(4,
             mkEmptyBT(),
             mkNodeBT(9, mkEmptyBT(), mkEmptyBT())),
           mkNodeBT(32,
             mkNodeBT(12, mkEmptyBT(), mkEmptyBT()),
             mkNodeBT(55,
               mkNodeBT(36,
                 mkNodeBT(34, mkEmptyBT(), mkEmptyBT()),
                 mkNodeBT(41, mkEmptyBT(), mkEmptyBT())),
               mkNodeBT(92,
                 mkEmptyBT(),
                 mkNodeBT(97, mkEmptyBT(), mkEmptyBT())))))

tKeys1Test = (tKeys1
              == insertListBST(keys1, mkEmptyBT()))
```

- Note that when the Python interpreter prints `tKeys1` it includes field names and values and has no line breaks.
- Answer 9 continued on next slide

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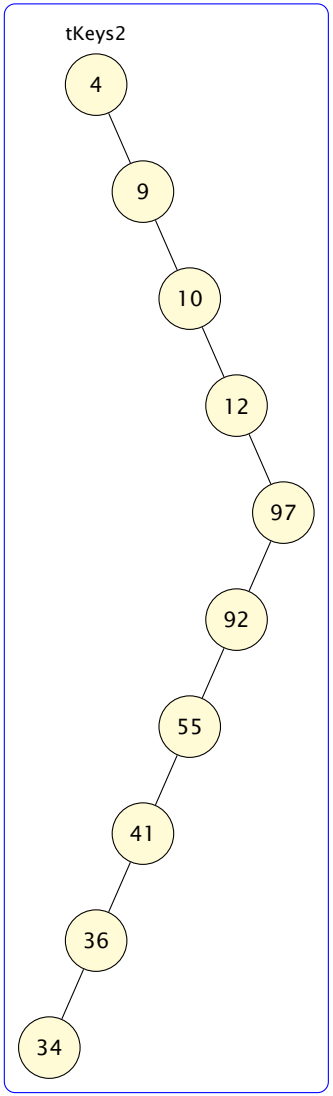
Answer 9 Insert List `tKeys2 = insertListBST(keys2, mkEmptyBT())`



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Binary Search Trees

Answer 9 Insert List `tKeys2 = insertListBST(keys2, mkEmptyBT())`

```
tKeys2 = mkNodeBT(4, mkEmptyBT(),
           mkNodeBT(9, mkEmptyBT(),
             mkNodeBT(10, mkEmptyBT(),
               mkNodeBT(12, mkEmptyBT(),
                 mkNodeBT(32, mkEmptyBT(),
                   mkNodeBT(97,
                     mkNodeBT(92,
                       mkNodeBT(55,
                         mkNodeBT(41,
                           mkNodeBT(36,
                             mkNodeBT(34, mkEmptyBT(),
                                          mkEmptyBT()),
                             mkEmptyBT()),
                           mkEmptyBT()),
                         mkEmptyBT()),
                       mkEmptyBT()),
                     mkEmptyBT()))))))

tKeys2Test = (tKeys2
              == insertListBST(keys2, mkEmptyBT()))
```

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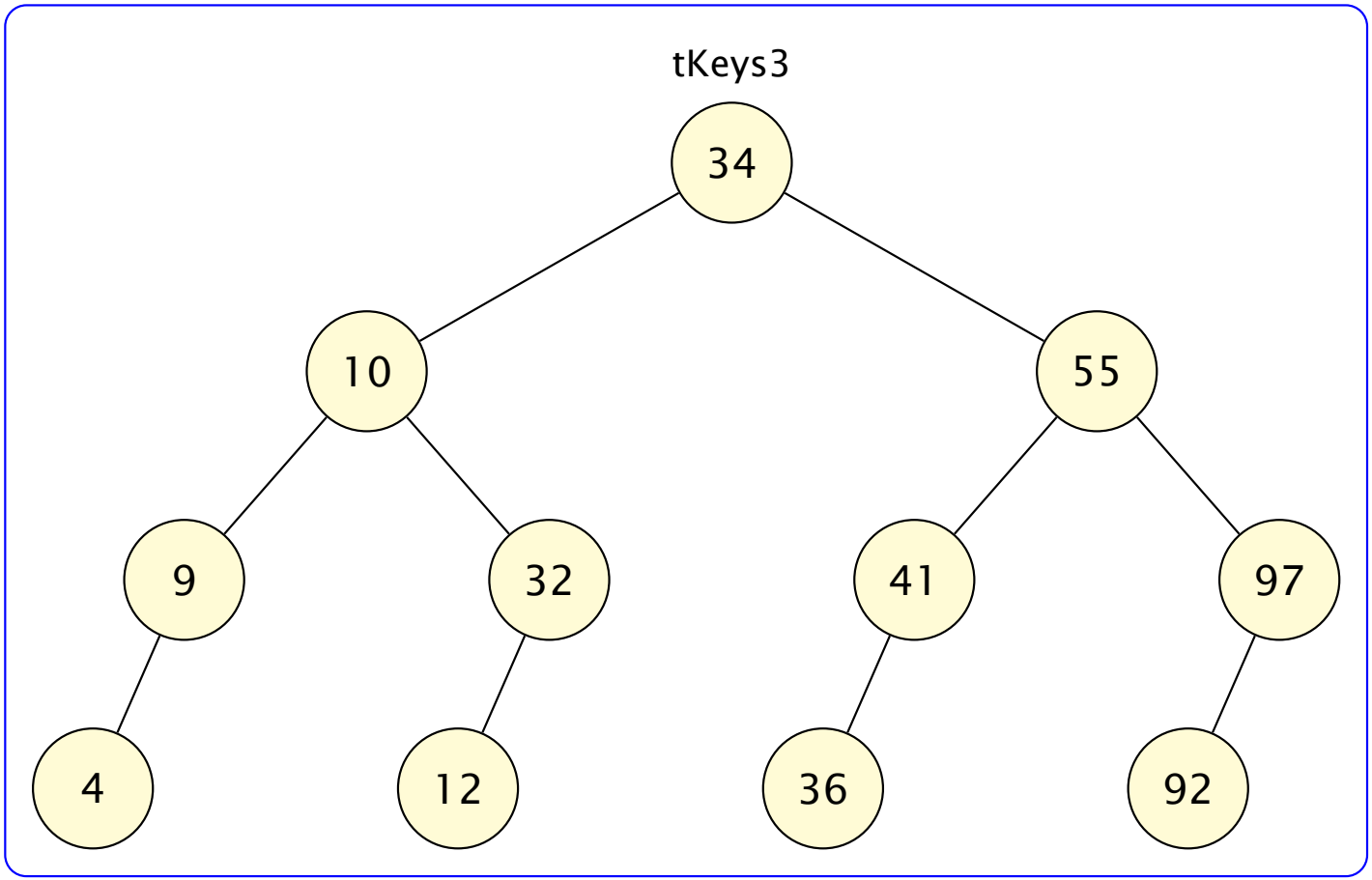
Binary Search Trees

Answer 9 Insert List `tKeys3 = insertListBST(keys3, mkEmptyBT())`



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Binary Search Trees

Answer 9 Insert List `tKeys3 = insertListBST(keys3, mkEmptyBT())`

```
tKeys3 = mkNodeBT(34,
           mkNodeBT(10,
             mkNodeBT(9,
               mkNodeBT(4, mkEmptyBT(), mkEmptyBT()),
               mkEmptyBT()),
             mkNodeBT(32,
               mkNodeBT(12, mkEmptyBT(), mkEmptyBT()),
               mkEmptyBT())),
           mkNodeBT(55,
             mkNodeBT(41,
               mkNodeBT(36, mkEmptyBT(), mkEmptyBT()),
               mkEmptyBT()),
             mkNodeBT(97,
               mkNodeBT(92, mkEmptyBT(), mkEmptyBT()),
               mkEmptyBT())))

tKeys3Test = (tKeys3
              == insertListBST(keys3, mkEmptyBT()))
```

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Binary Search Trees

Answer 9 Insert List

- Notice that tree `tKeys2` has height equal to the number of items 11
- The structure might as well be a list
- In this case the tree structure would not be more efficient than a list for searching.
- The height of `tKeys3` is 4 which is as compact a tree with the number of items between 8 and 15.
- `tKeys2` shows inserting a list in even partly sorted order results in the worst case for efficiency.
- If a binary search tree is built from insertion of a list of random data then it can be shown that the expected height of the tree is $O(\log n)$
- The proof of this requires knowledge of statistics outside the remit of this course — if interested, a proof is in Cormen et al. (2009, page 300)

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Binary Search Trees

Building a Compact BST

- To produce as compact a tree as possible, we could we could do the following:
- Sort the list
- Find the middle item in the sorted list
- Construct a binary tree node with the middle item as the data
- The left and right sub-trees should be formed by recursively building binary trees from the front and back parts of the sorted list
- Below is an implementation in Python

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Binary Search Trees

Building a Compact BST

```
604 def buildBST(xs) :
605   return bBST(mkEmptyBT(), sorted(xs))


608 def bBST(t,xs) :
609   if xs == [] :
610     return t
611   else :
612     half = len(xs) // 2
613     x = xs[half]
614     frontxs = xs[:half]
615     backxs = xs[half+1:]
616     if isEmptyBT(t) :
617       return (mkNodeBT(x,
618                 bBST(mkEmptyBT(),frontxs),
619                 bBST(mkEmptyBT(),backxs)))
620     else :
621       errMsg = ("bBST:␣Trying␣to␣insert" + str(xs)
622                 + "␣into␣nonempty␣tree" + str(t))
623       raise RuntimeError(errMsg)
```


Binary Trees

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Binary Search Trees

Building a Compact BST — Example

```
Python3>>> xs = [1, 9, 2, 8, 3, 7, 4, 6, 5]
Python3>>> buildBST(xs)
  NodeBT(dataBT=5,
    leftBT=NodeBT(dataBT=3,
             leftBT=NodeBT(dataBT=2,
                      leftBT=NodeBT(dataBT=1,
                               leftBT=EmptyBT(),
                               rightBT=EmptyBT()),
                      rightBT=EmptyBT()),
              rightBT=NodeBT(dataBT=4,
                        leftBT=EmptyBT(),
                        rightBT=EmptyBT())),
    rightBT=NodeBT(dataBT=8,
             leftBT=NodeBT(dataBT=7,
                      leftBT=NodeBT(dataBT=6,
                               leftBT=EmptyBT(),
                               rightBT=EmptyBT()),
                      rightBT=EmptyBT()),
             rightBT=NodeBT(dataBT=9,
                        leftBT=EmptyBT(),
                        rightBT=EmptyBT())))
Python3>>>
```

- Note that when the Python interpreter prints a namedtuple it includes field names and values and has no line breaks

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Binary Search Trees

Deleting a Node

- Deleting an item from a binary search tree involves more choices than insertion
- *Initial insight*
  - Find the node with the item (key) by following left or right sub trees
  - Delete the item by joining the two sub trees of the node
  - If the item is not in the tree, just return `mkEmptyBT()`
- The tricky bit is deciding how to join the two sub trees while keeping the binary search tree property and keeping the tree compact (otherwise we lose the advantage of a binary search tree).
- The following presents three alternatives which each use some information about binary search trees — each version is correct but the later versions produce a more compact tree.
- Below is the initial insight implemented in Python:

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Deleting an Item from a BST

Python

```
627 def deleteBST(x, t):
628   if isEmptyBT(t):
629     return mkEmptyBT()
630   else:
631     y = getDataBT(t)
632     leftT = getLeftBT(t)
633     rightT = getRightBT(t)
634     if x < y:
635       return mkNodeBT(y, (deleteBST(x,leftT)), rightT)
636     elif x > y:
637       return mkNodeBT(y, leftT, (deleteBST(x,rightT)))
638     else:
639       return joinBST(leftT, rightT)
```


Binary Trees

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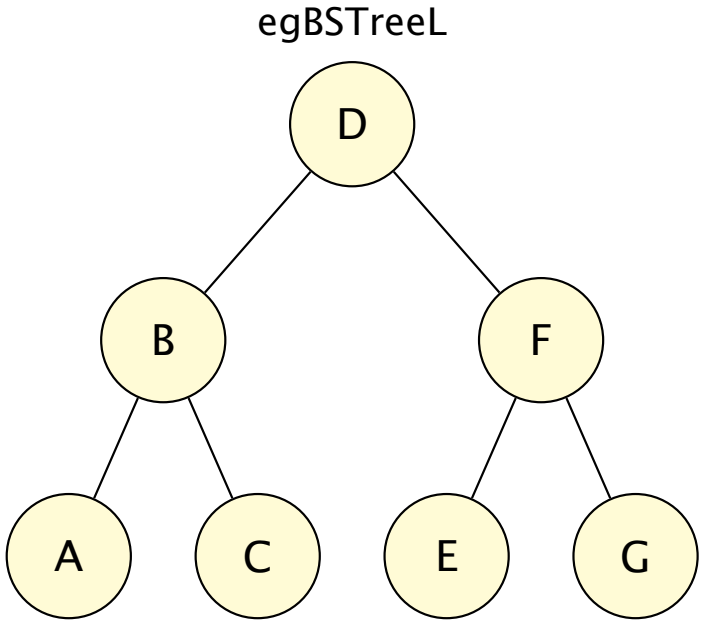
Example BST

- We now investigate different ways of joining two subtrees with the function `joinBST(leftT, rightT)`
- We shall use the small tree `egBSTreeL` to illustrate the choices deleting the node with key `D`



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Example BST

Python

- Here is a Python implementation of the tree egBSTreeL

```
122 egBSTreeL = mkNodeBT('D',
123                mkNodeBT('B',
124                  mkNodeBT('A',mkEmptyBT(),mkEmptyBT()),
125                  mkNodeBT('C',mkEmptyBT(),mkEmptyBT())
126                ),
127                mkNodeBT('F',
128                  mkNodeBT('E',mkEmptyBT(),mkEmptyBT()),
129                  mkNodeBT('G',mkEmptyBT(),mkEmptyBT())
130                ))
```


Binary Trees

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joinBST

Version 1

- `joinBST1` lists out all the elements of the left sub tree and inserts them in the right subtree.
- It does an in-order traversal of the left sub tree and then inserts the resulting list of items in the right subtree.

```
643 def joinBST1(leftT, rightT):
644   if isEmptyBT(leftT):
645     return rightT
646   else:
647     return (insertListBST(inOrderBT(leftT), rightT))
```


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Deleting an Item from a BST

Activity 10 joinBST Version 1

- Draw the diagram of the tree resulting from deleting D with `joinBST1`
- Why is `joinBST1` not a good strategy ?

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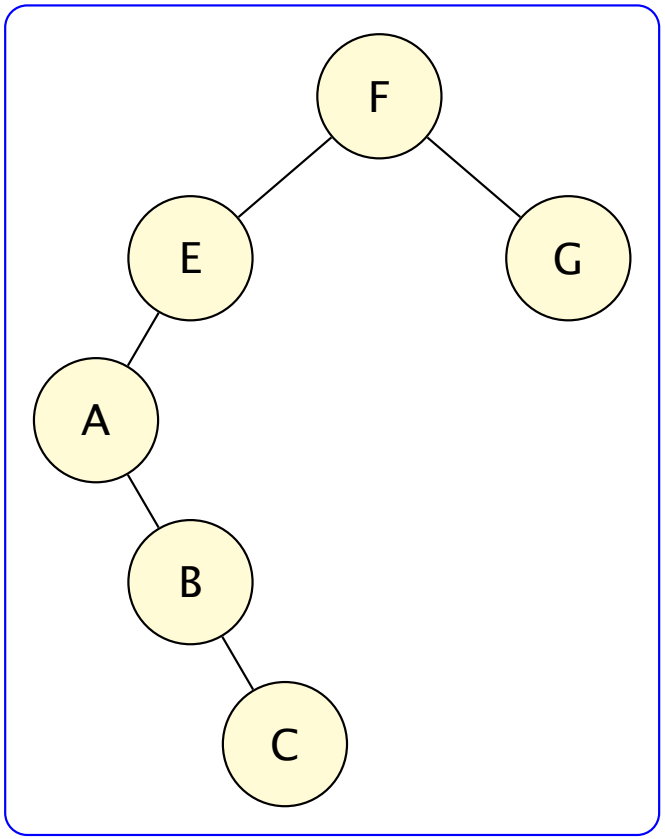
joinBST

Answer 10 joinBST Version 1 (a)



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joinBST

Answer 10 joinBST Version 1 (b)

```
651 delBSTreeJoin1 = joinBST1(egBSTreeLL,egBSTreeLR)

653 delBSTreeJoin1ans = \
654   NodeBT(dataBT='F',
655     leftBT=NodeBT(dataBT='E',
656               leftBT=NodeBT(dataBT='A',
657                         leftBT=EmptyBT(),
658                         rightBT=NodeBT(dataBT='B',
659                                   leftBT=EmptyBT(),
660                                   rightBT=NodeBT(dataBT='C',
661                                             leftBT=EmptyBT(),
662                                             rightBT=EmptyBT()))),
663                rightBT=EmptyBT()),
664     rightBT=NodeBT(dataBT='G',
665                leftBT=EmptyBT(),
666                rightBT=EmptyBT()))

668 delBSTreeJoin1test = delBSTreeJoin1 == delBSTreeJoin1ans
```

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joinBST

Version 2

- `joinBST1` results in a near linear structure and is not as compact as it could be.
- The first definition made no use of our knowledge of binary search trees.
- We know that:

```
maxKey leftT < minKey rightT
```

  since they were subtrees of the original Binary Search tree, `egBSTreeL`
- In particular we therefore know that the root of the left subtree is less than any item in the right subtree.
- So we attach the left subtree under the smallest element in the right sub tree.

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joinBST

Version 2

```
672 def joinBST2(leftT, rightT):
673   if isEmptyBT(rightT):
674     return leftT
675   else:
676     return (mkNodeBT(getDataBT(rightT),
677                      joinBST2(leftT, getLeftBT(rightT)),
678                      getRightBT(rightT)))
```


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Deleting an Item from a BST

Activity 11 joinBST Version 2

- Draw the diagram of the tree resulting fron deleting `D` with `joinBST2`
- Can you see how we can improve on `joinBST2` ?

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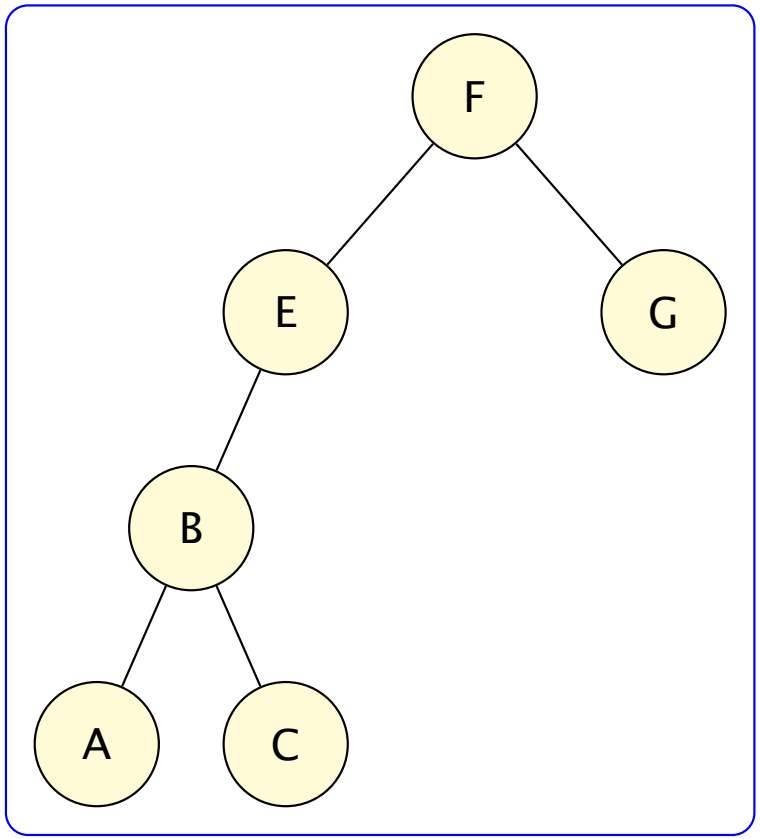
Deleting an Item from a BST

Answer 11 joinBST Version 2



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Deleting an Item from a BST

Answer 11 joinBST Version 2

```
681 delBSTreeJoin2 = joinBST2(egBSTreeLL,egBSTreeLR)

683 delBSTreeJoin2ans = NodeBT('F',
684                         NodeBT('E',
685                           NodeBT('B',
686                             NodeBT('A', EmptyBT(), EmptyBT()),
687                             NodeBT('C', EmptyBT(), EmptyBT())),
688                           EmptyBT()),
689                         NodeBT('G', EmptyBT(), EmptyBT()))

691 delBSTreeJoin2test = (delBSTreeJoin2 == delBSTreeJoin2ans)
```

- To see how this works we shall do a step by step evaluation
- Follow the function code for `joinBST2` from line 672 on slide 142
- Note from the definitions of `delBSTreeJoin1test` (from line 668 on slide 140) and `delBSTreeJoin2test` (from line 691 on slide 145) we can use the field names or leave them out
- Answer 11 continued on next slide

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Deleting an Item from a BST

Answer 11 joinBST Version 2

- **Step 1** In the first call to `joinBST2` the `leftT` is the tree rooted at `B` and the `rightT` is the tree rooted at `F`
- Line `673` tests if the second argument to `joinBST2` is an empty tree
- Since it is not empty, `joinBST2` evaluates to the value at line `676`
- Hence we have

```
mkNodeBT('F',
         joinBST2(leftT, getLeftBT(rightT)),
         getRightBT(rightT))
```

- Answer 11 continued on next slide

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Answer 11 joinBST Version 2

- **Step 2** Since the return value has a recursive call to `joinBST2` we need to evaluate that.
- Its second argument is `rightT.leftBT` which is the tree rooted at `E`
- Line `673` tests if the first argument to `joinBST2` is an empty tree
- Since it is not empty, `joinBST2` evaluates to the value at line `676`
- Hence we have

```
mkNodeBT('F',
         mkNodeBT('E',
                  joinBST2(leftT, getLeftBT(getLeftBT(rightT))),
                  getRightBT(getLeftBT(rightT))),
         getRightBT(rightT))
```

- Answer 11 continued on next slide

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Deleting an Item from a BST

Answer 11 joinBST Version 2

- **Step 3** We have to evaluate a further recursive call
- The second argument to the recursive call to `joinBST2` is `rightT.leftBT.leftBT` which is `EmptyBT()`, so the recursive call to `joinBST2` evaluates to `leftT`
- `rightT.leftBT.rightBT` evaluates to `EmptyBT()`
- Hence we have

```
mkNodeBT('F',
         mkNodeBT('E',
                  leftT,
                  mkEmptyBT()),
         getRightBT(rightT))
```

- Answer 11 continued on next slide

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Deleting an Item from a BST

Answer 11 joinBST Version 2

- Hence the final value is

```
NodeBT('F',
  NodeBT('E',
    NodeBT('B',
      NodeBT('A', EmptyBT(), EmptyBT()),
      NodeBT('C', EmptyBT(), EmptyBT())),
    EmptyBT()),
  NodeBT('G', EmptyBT(), EmptyBT()))
```

- Doing a step by step evaluation of recursive function calls should help you get a better feel for recursive thinking.
- We can do better than this — see the following.

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joinBST

Final Version

- We can make better use of our knowledge of Binary Search trees
- We know that:

```
maxKey leftT < root key < minKey rightT
```

- Hence we can promote the minimum item in the right subtree to be the new root and delete it from its original position.
- **Note** we could equally well promote the maximum item in the left subtree to be the new root (and delete it from its original position).

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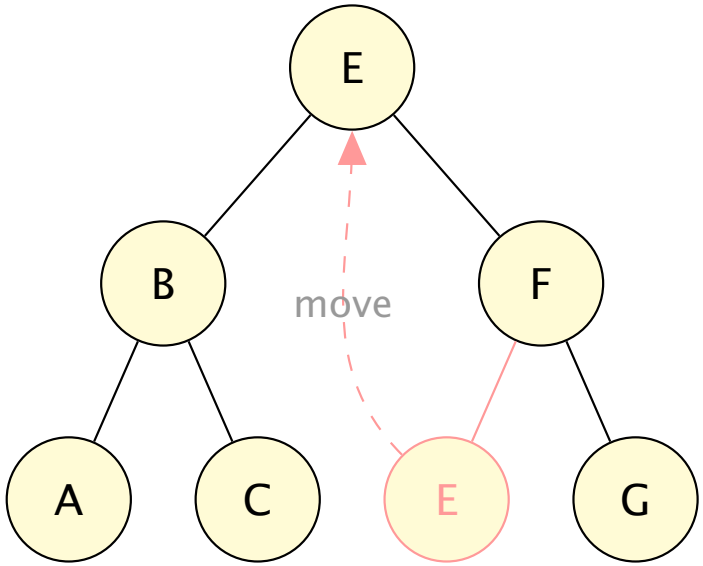
joinBST

Final Version



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joinBST

Final Version

- Here is Python code for the above diagram:

```
696 def joinBST(leftT, rightT):
697   if isEmptyBT(rightT):
698     return leftT
699   else:
700     (y,t) = splitBST(rightT)
701     return mkNodeBT(y, leftT, t)
```

- `splitBST` will take the right subtree and return a pair of minimum item and the subtree with that item removed.
- This preserves much of the compact nature of the binary search tree.

Binary Trees

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splitBST

Version 1

- We still have choices in defining `splitBST`
- We could define `splitBST` by finding the minimum item and then deleting that from the subtree.

```
706 def splitBST1(t):
707   if isEmptyBT(t):
708     raise RuntimeError("splitBST1 applied to EmptyBT()")
709   elif isEmptyBT(getLeftBT(t)):
710     return (getDataBT(t), getRightBT(t))
711   else:
712     y = minItemBST(getLeftBT(t))
713     return (y, mkNodeBT(getDataBT(t),
714                         deleteBST(y, getLeftBT(t)),
715                         getRightBT(t)))

717 def minItemBST(t):
718   if isEmptyBT(t):
719     raise RuntimeError("minItemBST applied to EmptyBT()")
720   elif isEmptyBT(getLeftBT(t)):
721     return (getDataBT(t))
722   else:
723     return (minItemBST(getLeftBT(t)))
```


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splitBST

Final Version

- We can do better than `splitBST1`
- It is possible to define `splitBST` using only one traversal of the tree.

```
727 def splitBST(t):
728   if isEmptyBT(t):
729     raise RuntimeError("splitBST applied to EmptyBT()")
730   else:
731     x = getDataBT(t)
732     t1 = getLeftBT(t)
733     t2 = getRightBT(t)
734     if isEmptyBT(t1):
735       return (x,t2)
736     else:
737       (y,t3) = splitBST(t1)
738       return (y, mkNodeBT(x, t3, t2))
```


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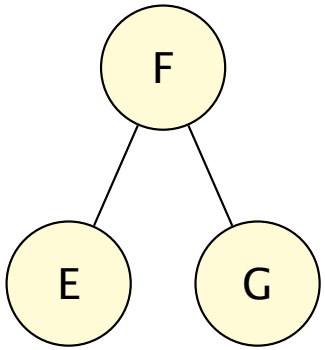
Binary Search Tree

Activity 12 Split Trace

- Trace an evaluation of `splitBST(egBSTreeLR)`

splitBST(egBSTreeLR)




```
160 egBSTreeLR = mkNodeBT('F',
161                   mkNodeBT('E',mkEmptyBT(),mkEmptyBT()),
162                   mkNodeBT('G',mkEmptyBT(),mkEmptyBT())
163                 )
```

- egBSTreeLR is defined at line 160 on slide 155, splitBST is defined at line 727 on slide 154

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Binary Search Tree

Answer 12 Split Trace

- Evaluation of `splitBST(egBSTreeLR)`

Step 1

`getLeftBT(egBSTreeLR)` is not empty so the `else` clause at line `736` is executed

Step 2

A recursive call is made to `splitBST` with argument `getLeftBT(egBSTreeLR)`

`getLeftBT(getLeftBT(egBSTreeLR))` is empty so the `if` at line `734` returns

`('E',getRightBT(getLeftBT(egBSTreeLR)))` which is `('E',EmptyBT())`

- Answer 12 continued on next slide

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Binary Search Tree

Answer 12 Split Trace

Step 3

The calling function then returns

```
('E', makeBT('F', EmptyBT(),
getRightBT(egBSTreeLR)))
```

```
('E',NodeBT('F',
       EmptyBT(),
       NodeBT('G',EmptyBT(),EmptyBT())))
```

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Binary Trees

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Binary Search Tree

Activity 13 Join Trace

- Trace an evaluation of

```
joinBST(egBSTreeLL,egBSTreeLR)
```

- egBSTreeLL is defined at line 146 on slide 60, joinBST is defined at line 696 on slide 152

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Binary Search Tree

Answer 13 Join Trace

- Evaluation of

```
joinBST(egBSTreeLL,egBSTreeLR)
```

Step 1

The second argument to `joinBST` is not the empty tree so the `else` clause at line `699` — this invokes `splitBST(egBSTreeLR)`

Step 2

From the previous activity, `splitBST(egBSTreeLR)` returns

```
('E', makeBT('F', EmptyBT(),
getRightBT(egBSTreeLR)))
```

- Answer 13 continued on next slide

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Binary Search Tree

Answer 13 Join Trace

Step 3

Finally, the `return` statement returns

```
NodeBT('E',
  NodeBT('B',
    NodeBT('A', EmptyBT(), EmptyBT()),
    NodeBT('C', EmptyBT(), EmptyBT())),
  NodeBT('F',
    EmptyBT(),
    NodeBT('G', EmptyBT(), EmptyBT())))
```

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Binary Search Tree

Activity 14 Delete Trace

- Trace an evaluation of

```
deleteBST('D',egBSTreeL)
```

- egBSTreeL is defined at line 122 on page 59, deleteBST is defined at line 627 on page 134

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Binary Search Tree

Answer 14 Delete Trace

- Evaluation of `deleteBST('D',egBSTreeL)`

Step 1

The second argument of `deleteBST` is not the empty tree so the `else` clause at line `630` is executed.

Step 2

The first argument of `deleteBST` is `'D'` which is equal to the item at the root of the tree which is the second argument, so the `else` clause at line `638` is executed

Step 3

This evaluates to `joinBST(egBSTreeLL,egBSTreeLR)` — see previous activity

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Binary Search Trees

Performance

- As we noted earlier, on average the height of a binary search tree is $O(\log n)$ where n is the number of nodes in the tree.
- However in the worst case the height is $O(n)$ and this will affect the performance of searches.
- However it is possible to construct variants of binary search trees which have $O(\log n)$ performance in both average and worst cases
- In the next section we will consider one approach.

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Commentary 4

Height Balanced Trees (AVL Trees)

4 Height Balanced Trees

- Binary search trees with the *height balanced* property
- Also called AVL trees
- Inserting a node
- AVL transformations
- Local changes preserve global AVL property
- Deletion
- AVL trees application: representing sets (advanced topic)
- Note: Haskell uses the same ideas but with *size* balanced trees
- Python uses something like dictionaries to implement sets using hashtables

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Height Balanced Trees

Introduction

- Binary search trees have the problem that in the worst case the complexity of a search could be $O(n)$ and maintaining a complete tree during insertions and deletions involves too much restructuring.
- A solution is to keep the tree *balanced* so that access time is still $O(\log n)$ in both the average and worst cases.
- The essential approach is to have some *local* transformations involving only a few nodes to keep the tree *height balanced*.
- We shall consider one approach called *AVL Trees*, named after the Russian inventors G M Adelson-Velskii and E M Landis
- *AVL Trees* are Binary Search Trees with the property that for every subtree the heights of the trees differs by at most 1 (the *balance factor*).
- AVL trees require an extra couple of functions to maintain the AVL property on each insertion or deletion.

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AVL Trees

Data Type

- As with the Binary Search Tree, we shall use a union of named tuples to represent the data type for an AVL Tree.
- Note that we store the height of a tree in the node
- This is essential to avoid lots of tree traversals to re-calculate balance factors

```
743 # AVL Tree Data Type

745 # from collections import namedtuple

747 EmptyABT = namedtuple('EmptyABT',[])

749 NodeABT = (namedtuple('NodeABT',
750             ['heightABT','dataABT','leftABT','rightABT']))

752 # Tree type --- Augmented Binary Tree

754 # from typing import TypeVar,Union,NewType

756 # T = TypeVar('T')
757 ABTree = NewType('ABTree',Union[EmptyABT,NodeABT]).
```


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AVL Trees

Operations(1)

```
759 # AVL Tree Operations

761 def mkEmptyABT() -> ABTree :
762   return EmptyABT()

764 def mkNodeABT(x: T,t1: ABTree,t2: ABTree) -> ABTree :
765   h = 1 + max(getHeightABT(t1),getHeightABT(t2))
766   return NodeABT(h,x,t1,t2)

768 def isEmptyABT(t: ABTree) -> bool :
769   return t == EmptyABT()
```


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AVL Trees

Operations (2)

```
771 def getHeightABT(t: ABTree) -> int :
772   if isEmptyABT(t) :
773     return 0
774   else:
775     return t.heightABT

777 def getDataABT(t: ABTree) -> T :
778   if isEmptyABT(t) :
779     raise RuntimeError("getDataABT applied to EmptyABT()")
780   else:
781     return t.dataABT

783 def getLeftABT(t: ABTree) -> ABTree :
784   if isEmptyABT(t) :
785     raise RuntimeError("getLeftABT applied to EmptyABT()")
786   else:
787     return t.leftABT

789 def getRightABT(t: ABTree) -> ABTree :
790   if isEmptyABT(t) :
791     raise RuntimeError("getRightABT applied to EmptyABT()")
792   else:
793     return t.rightABT
```


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AVL Trees

Property Functions (1)

```
797 def isBSABTree(t):
798   return orderedList(inOrderABT(t))

800 def inOrderABT(t):
801   if isEmptyABT(t):
802     return []
803   else:
804     return (inOrderABT(getLeftABT(t)) + [getDataABT(t)]
805             + inOrderABT(getRightABT(t)))
```

```
832 def convertBTtoABT(t):
833   if isEmptyBT(t):
834     return mkEmptyABT()
835   else:
836     leftABT = convertBTtoABT(getLeftBT(t))
837     rightABT = convertBTtoABT(getRightBT(t))
838     return mkNodeABT(getDataBT(t), leftABT, rightABT)
```


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AVL Trees

Property Functions (2)

```
809 def balFactorABT(t):
810   if isEmptyABT(t):
811     return 0
812   else:
813     return (getHeightABT(getLeftABT(t))
814             - getHeightABT(getRightABT(t)))
```

```
818 def hasAVLpropABT(t):
819   if isEmptyABT(t):
820     return True
821   else:
822     return (abs(balFactorABT(t)) <= 1
823             and hasAVLpropABT(getLeftABT(t))
824             and hasAVLpropABT(getRightABT(t)))
```

```
827 def isAVLABTree(t):
828   return (isBSABTree(t) and hasAVLpropABT(t))
```


Binary Trees

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AVL Trees

Health Warning

- Some texts define the height of a singleton node to be zero — just subtract one from the height as defined here.
- Some texts do not use empty trees — so where these notes might say a singleton nodes has an element and two empty subtrees, some texts might say a singleton node has no subtrees
- Some texts define the height of a subtree differently to the height of a tree or define a subtree differently to here.
- Some texts define the balance factor as the absolute value or the height of the right sub tree minus the height of the left sub tree
- In all cases be aware that you have choices in the exact definition of some terms but the ideas will be the same.

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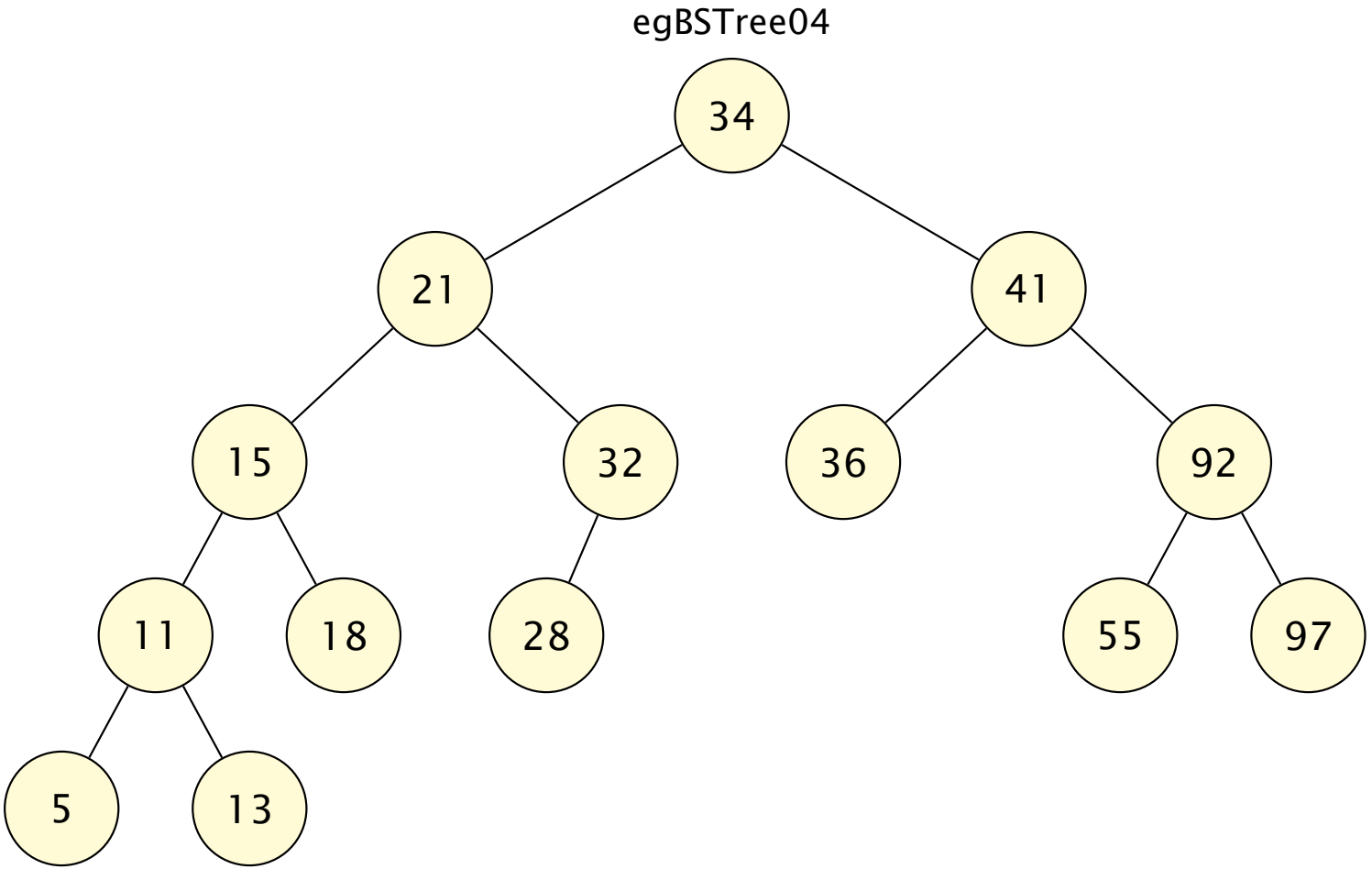
Binary Search Tree

Activity 15 Heights and Balance Factors

- For the following diagram of a binary search tree, egBSTree04, add the height and balance factor for each node.



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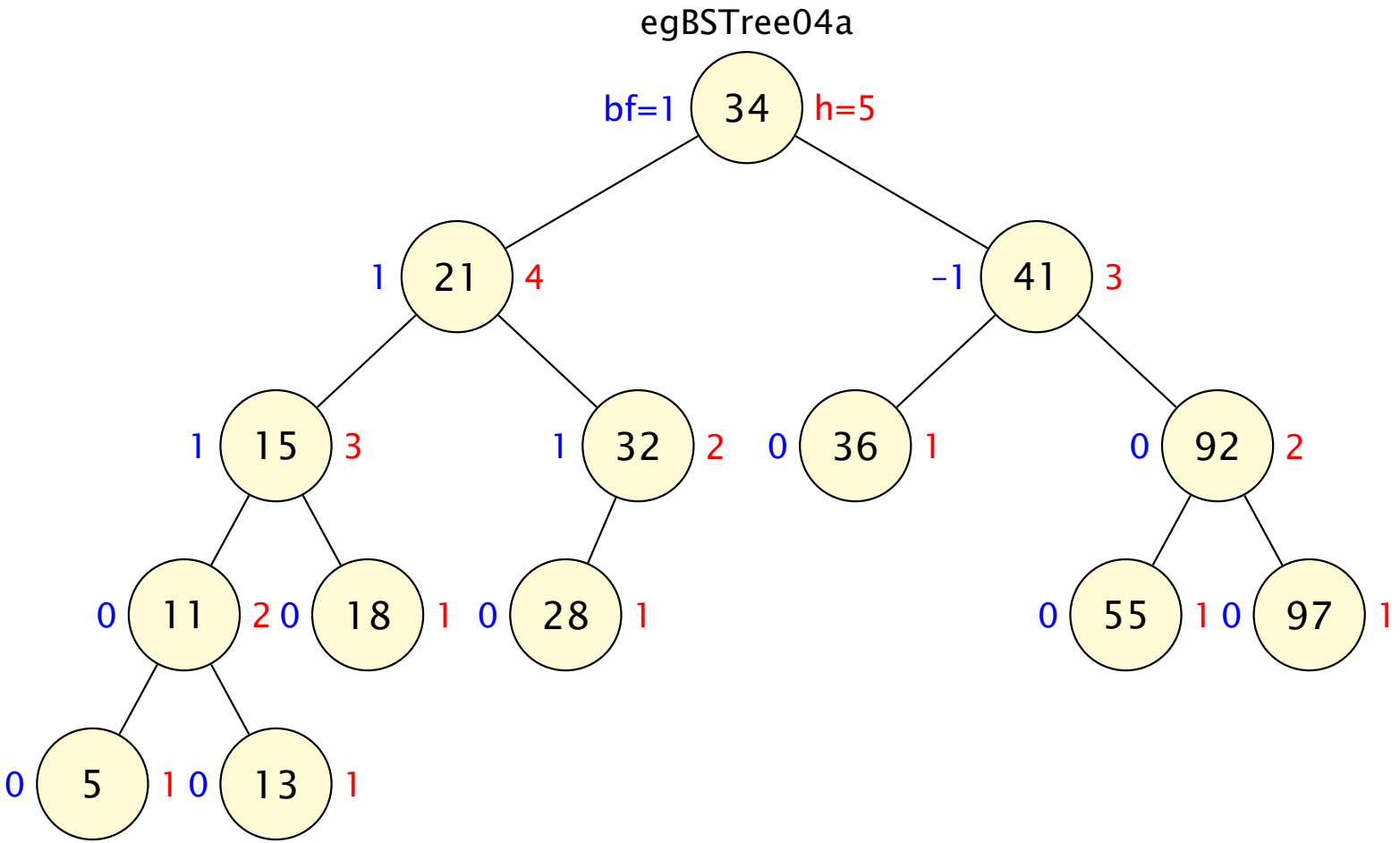
Binary Search Tree

Answer 15 Heights and Balance Factors



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Heights and Balance Factors

Activity 16 Add Item LL

- Add the item with key 7 to the tree, egNSTree04, and recalculate the heights and balance factors
- Identify the lowest node in the tree which is out of balance.

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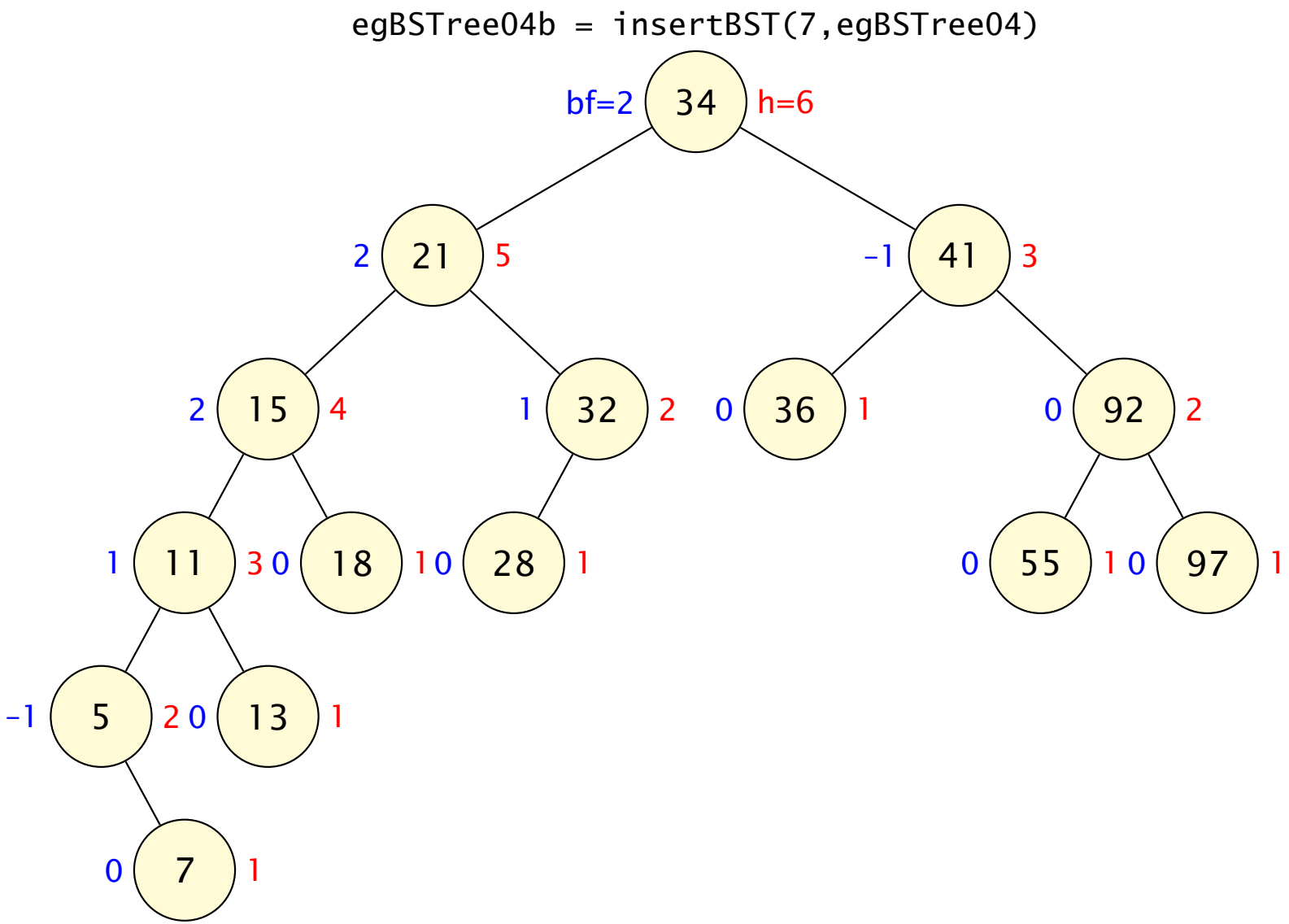
Heights and Balance Factors

Answer 16 Add Item LL



- Lowest node which is out of balance is node with key 15

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AVL Tree Transformations

Sample Tree

- Since the subtree of `egBSTree04b = insertBST(7,egBSTree04)` at node with key 15 is the part of the tree out of balance we shall focus on that

```
NodeABT(4, 15,
  NodeABT(3, 11,
    NodeABT(2, 5,
      EmptyABT(),
      NodeABT(1, 7, EmptyABT(), EmptyABT())),
    NodeABT(1, 13, EmptyABT(), EmptyABT())),
  NodeABT(1, 18, EmptyABT(), EmptyABT()))
```


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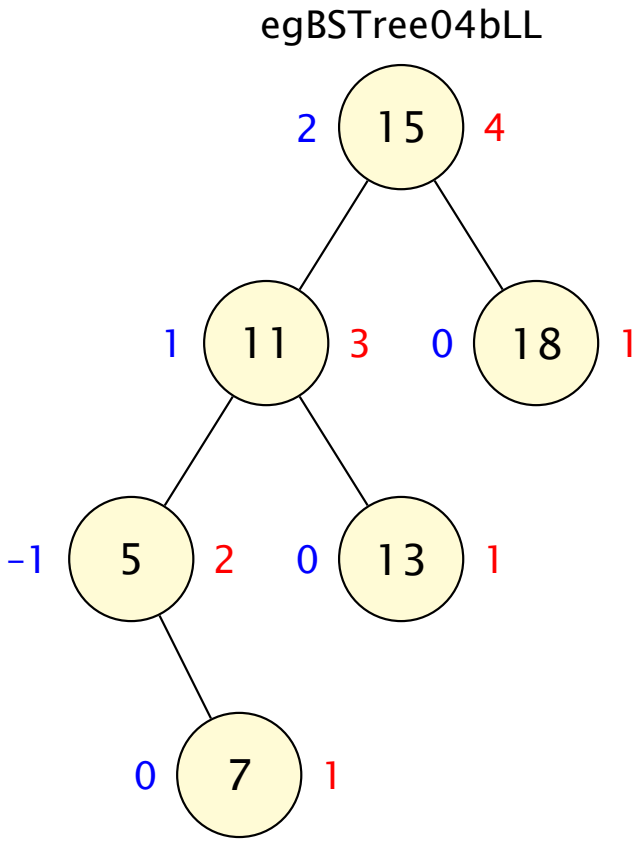
AVL Tree Transformation

Sample Tree



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AVL Tree Transformations

Example Transformation

- We can make this tree balanced by
- Make the subtree with root 15 the right child of 11
- Make the subtree with root 13 the left child of 15
- Leave the subtree with root 5 as the left child of 11
- Make the new subtree with root 11 the child of wherever the original subtree with root 15 was (the left child of 21)
- This results in the following tree.

```
NodeABT(3, 11,
  NodeABT(2, 5,
    EmptyABT(),
    NodeABT(1, 7, EmptyABT(), EmptyABT())),
  NodeABT(2, 15,
    NodeABT(1, 13, EmptyABT(), EmptyABT()),
    NodeABT(1, 18, EmptyABT(), EmptyABT())))
```


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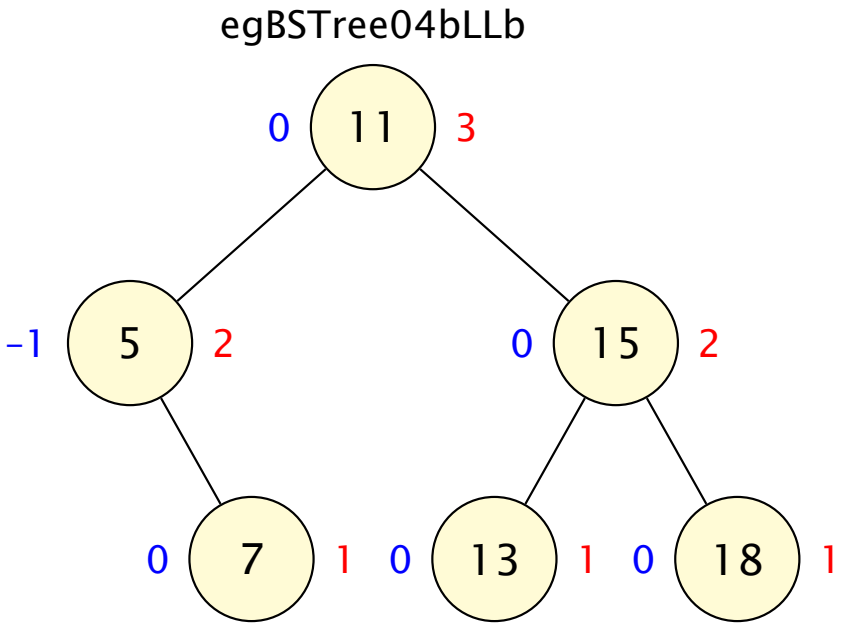
AVL Tree Transformation

Sample Tree — Transformed



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AVL Tree Transformations

Python

- This transformation is an instance of what is called a *right rotation*
- Here is Python code that implements it.

```
843 def rotr(t):
844   k = getDataABT(t)
845   kL = getDataABT(getLeftABT(t))
846   leftLeftT = getLeftABT(getLeftABT(t))
847   leftRightT = getRightABT(getLeftABT(t))
848   rightT = getRightABT(t)
849   return (mkNodeABT(kL,
850                     leftLeftT,
851                     mkNodeABT(k, leftRightT, rightT)))
```


Binary Trees

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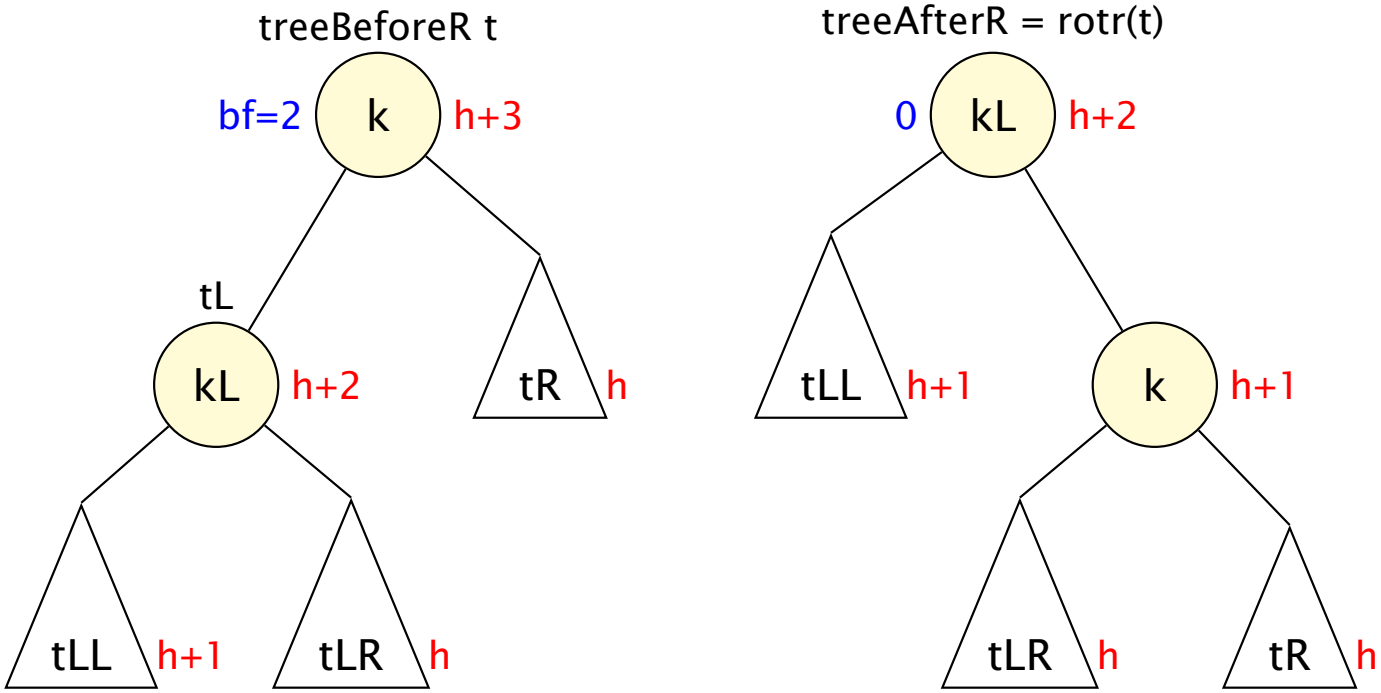
Right Rotation

Diagram tree `t` to tree `rotr(t)`



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Height and Balance Factors

Activity 17 Add Item RR

- Consider egBSTree04 again (defined in Activity 15 on slide 172) — now add node with key 96 and recalculate the heights and balance factors

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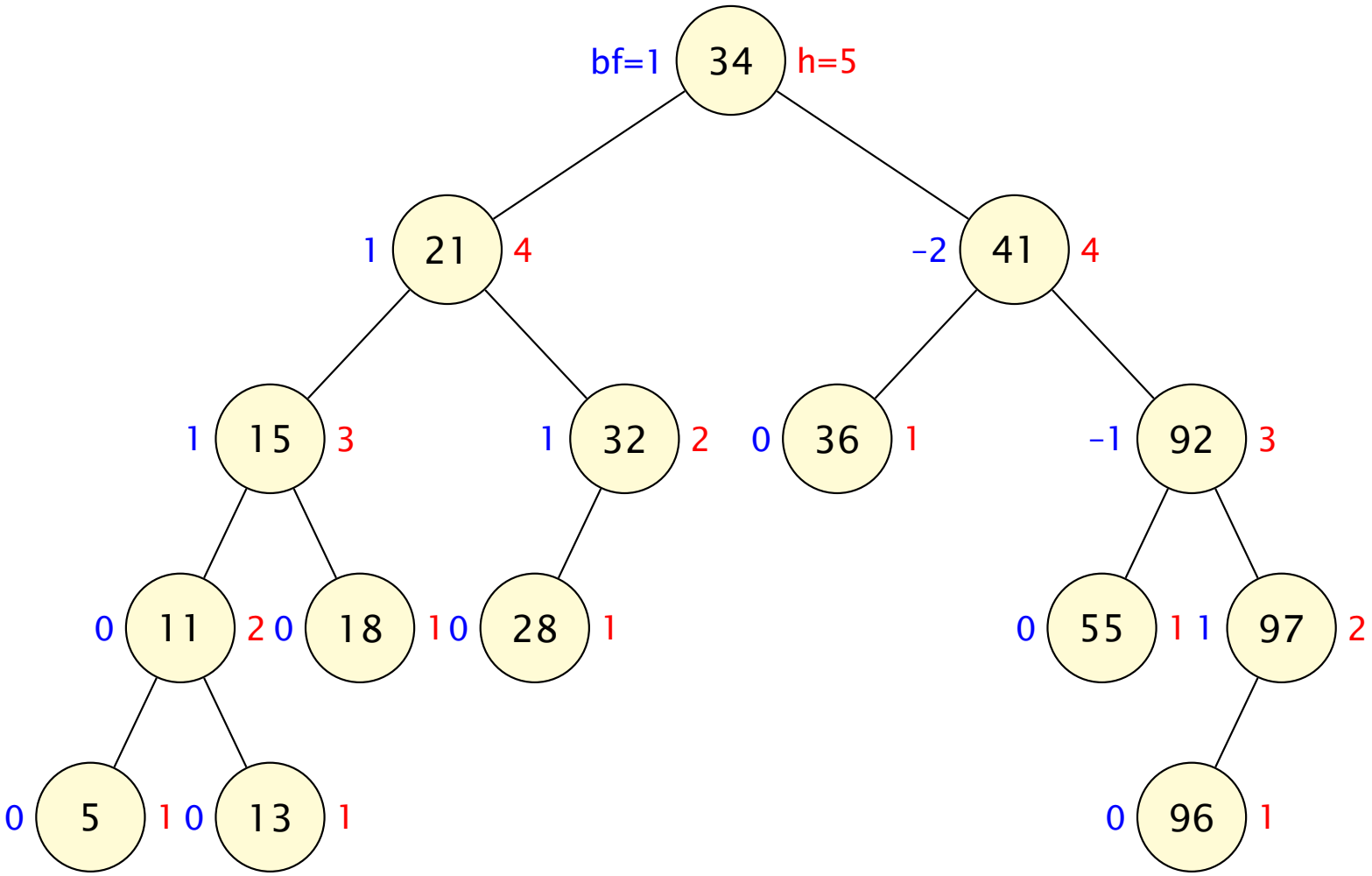
Height and Balance Factors

Answer 17 Add Item RR

egBSTree04c = insertBST(96,egBSTree04)



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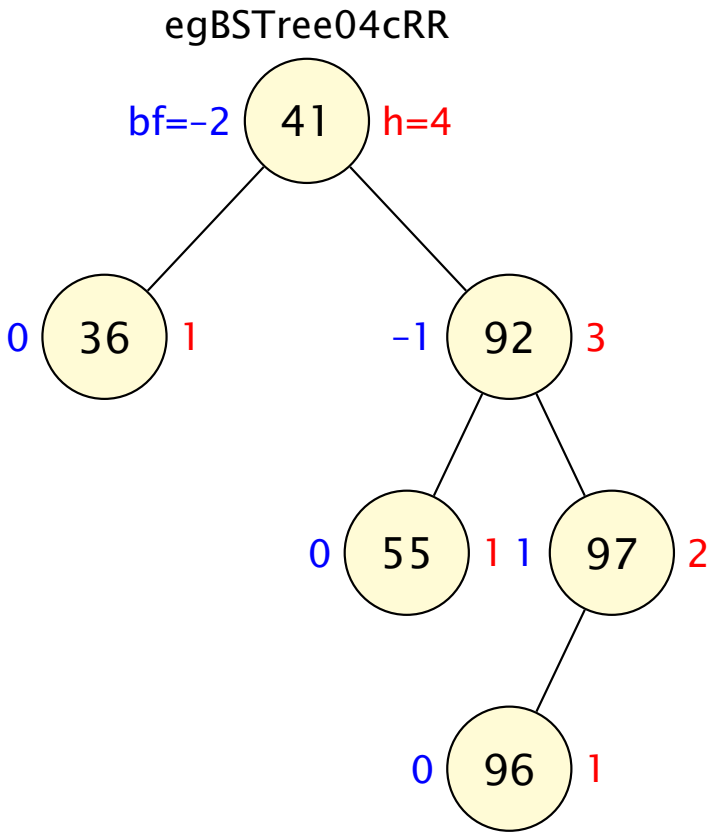
AVL Tree Transformations

Example Tree RR

- The subtree at node 41 is now unbalanced with the addition of the node with key 96 to the right subtree of the right subtree.



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AVL Tree Transformations

Activity 18 Rebalance RR

- This is similar to the previous example but on the right side
- Describe how this can be rebalanced using a mirror image local transformation.

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AVL Tree Transformations

Answer 18 Rebalance RR

- We can make this tree balanced by:
- Make the subtree with root 41 the left child of 92
- Make the subtree with root 55 the right child of 41
- Leave the subtree with root 97 as the right child of 92
- Make the new subtree with root 92 the child of wherever the original subtree with root 41 was (the right child of 34)

```
NodeABT(3, 92,
  NodeABT(2, 41,
    NodeABT(1, 36, EmptyABT(), EmptyABT()),
    NodeABT(1, 55, EmptyABT(), EmptyABT())),
  NodeABT(2, 97,
    NodeABT(1, 96, EmptyABT(), EmptyABT()),
    EmptyABT()))
```

- Answer 18 continued on next slide

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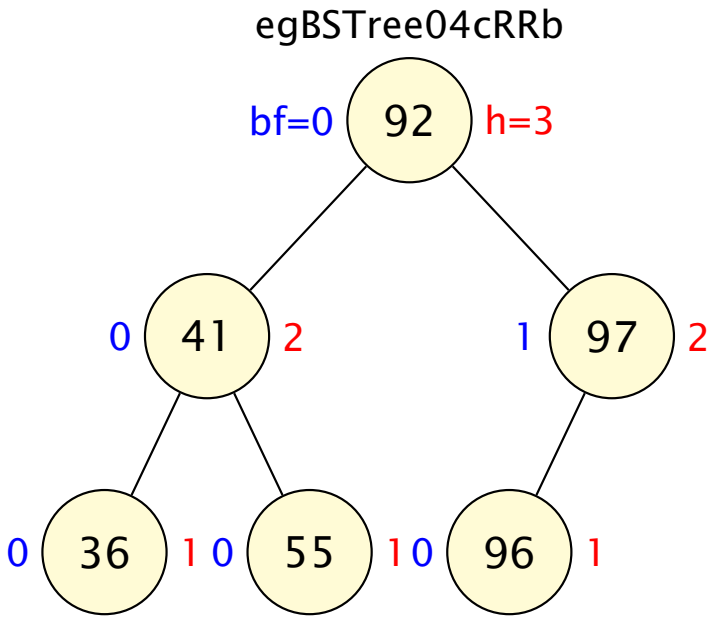
Answer 18 Rebalance RR



- The transformation is called a *left rotation*

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AVL Tree Transformations

Left Rotation — Python

- The transformation (given in the answer) is an instance of what is called a *left rotation*
- Here is the Python code that implements it.

```
853 def rotl(t):
854   k = getDataABT(t)
855   kR = getDataABT(getRightABT(t))
856   rightLeftT = getLeftABT(getRightABT(t))
857   rightRightT = getRightABT(getRightABT(t))
858   leftT = t.leftABT
859   return (mkNodeABT(kR,
860                     mkNodeABT(k, leftT, rightLeftT),
861                     rightRightT))
```

- This is a mirror image of the *right rotation* as you can see from the two diagrams describing it below.

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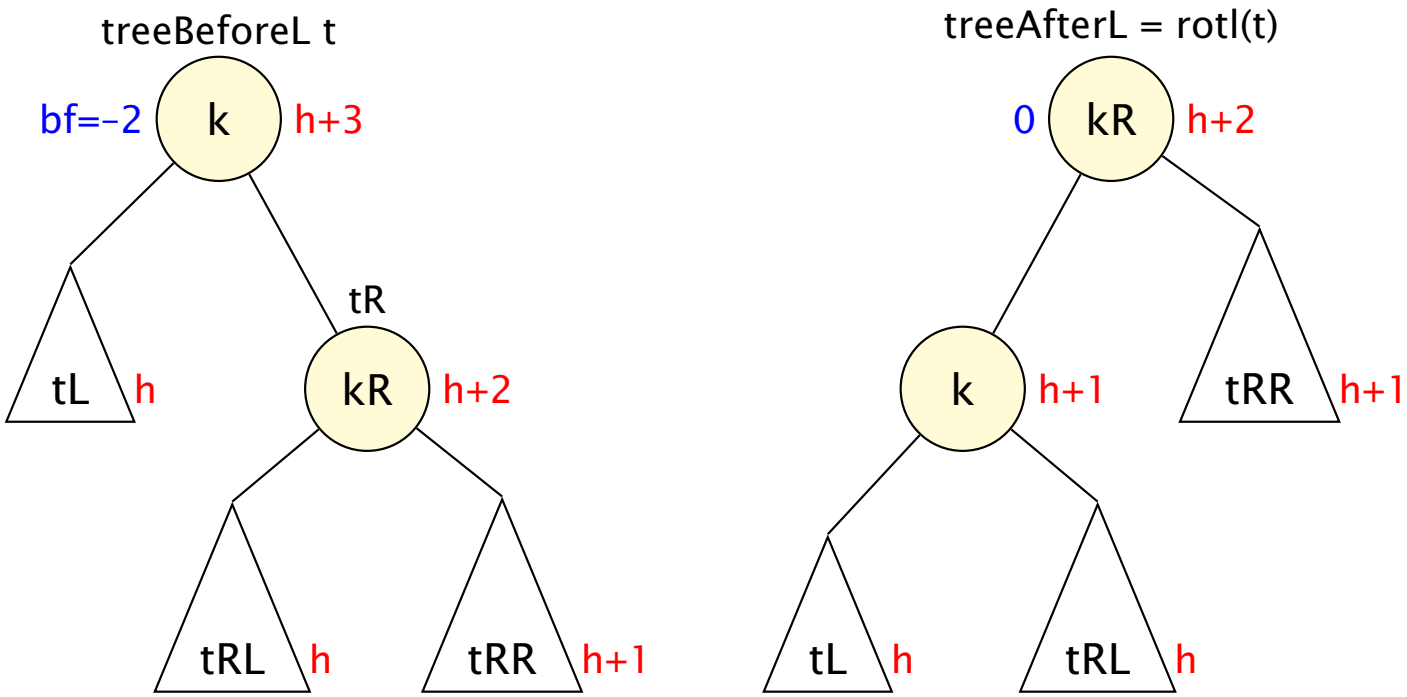
Left Rotation



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AVL Tree Transformations

Insight

- The functions for insertion and deletion of an item in an AVL tree will be the same as a Binary Search tree except
- When we construct a new tree we must maintain the AVL property via a function `makeAVLTree` (line 865 on slide 210) not just `makeABTree` (line 764 on slide 167). (some texts call this rebalancing or something similar)
- What we know is that the original tree must be a properly formed AVL tree and that the insertion or deletion of one item can alter the height of any subtree by at most 1.
- Hence we can implement `makeAVLTree(x, leftT, rightT)` assuming that `leftT` and `rightT` are both AVL trees whose heights differ by at most 2.
- We proceed by analysing each possible case and provide a manipulation of the tree for each case. Consider the diagram below:

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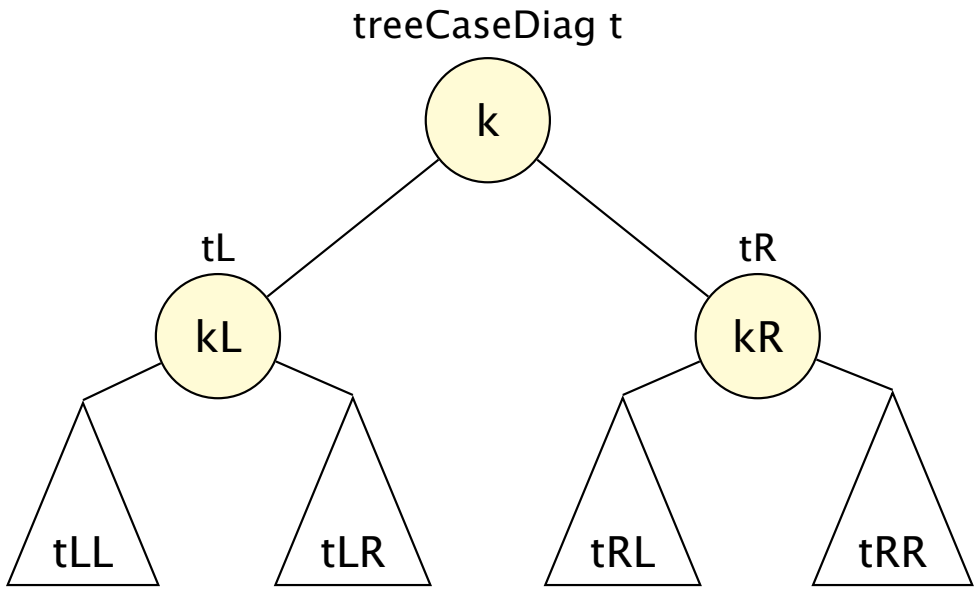
Cases Diagram



- Our right and left rotation functions, `rotr` and `rotl` have dealt with the cases where the subsubtrees LL and RR had increased by one caused the balance to go outside the permitted range.
- We now have to investigate cases where the subsubtrees LR or RL become heavy.
- Below is an example, `egBSTree05`

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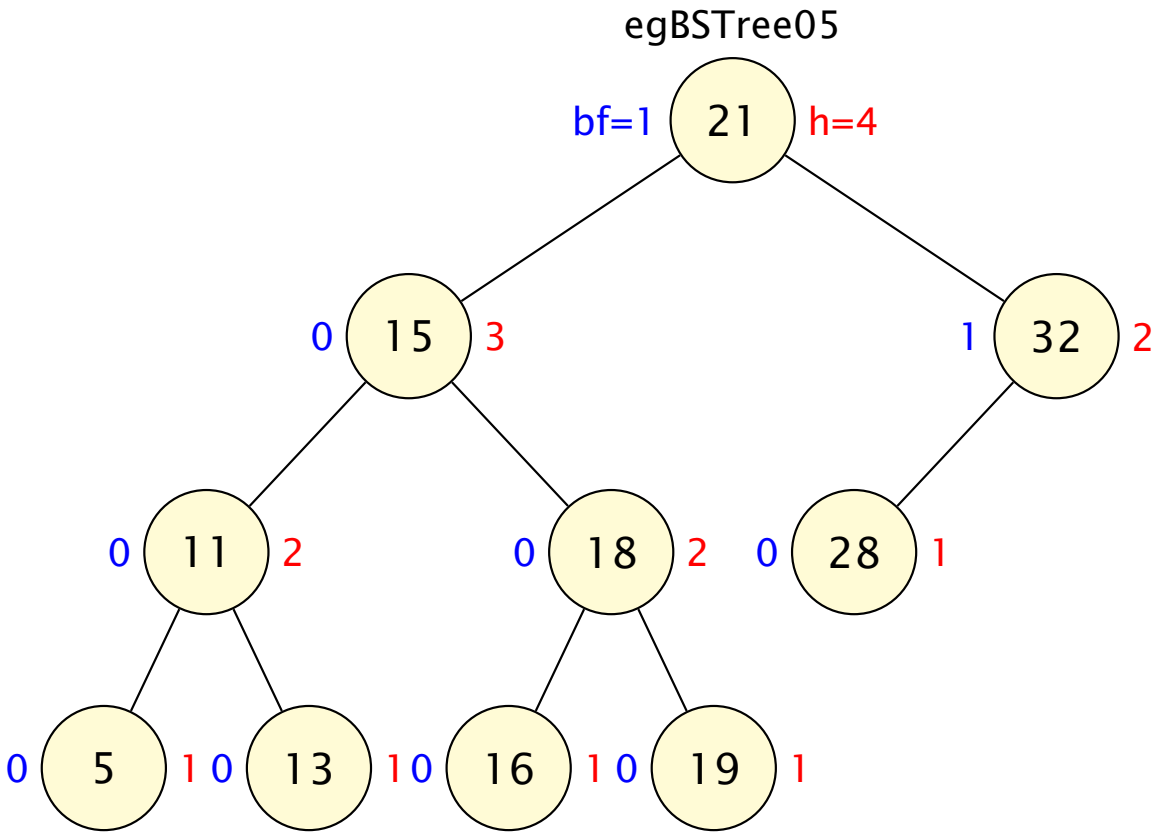
Tree egBSTree05



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Activity 19 egBSTree05 Add Item LR 1

- Add the item with key 20 to the tree and recalculate the heights and balance factors
- Identify the lowest node in the tree which is out of balance.

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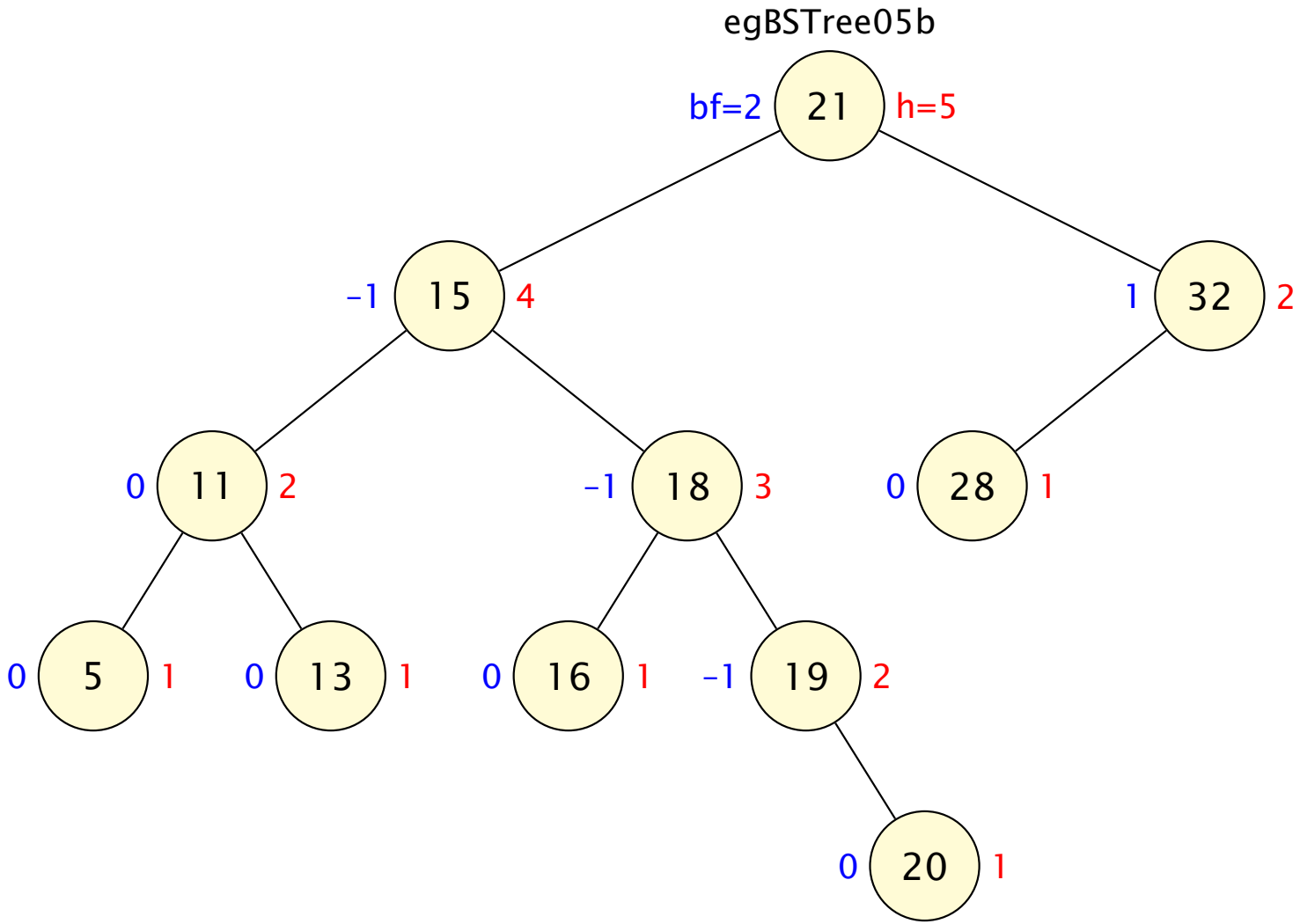
Answer 19 egBSTree05 Add Item LR 1



- The node with key 21 has balance factor 2 and is the lowest node out of balance.

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Activity 20 Add Item LR 2

- Given the resulting tree from Self-assessment activity 19, does a right rotation around the lowest node which is out of balance bring it back to balance ?

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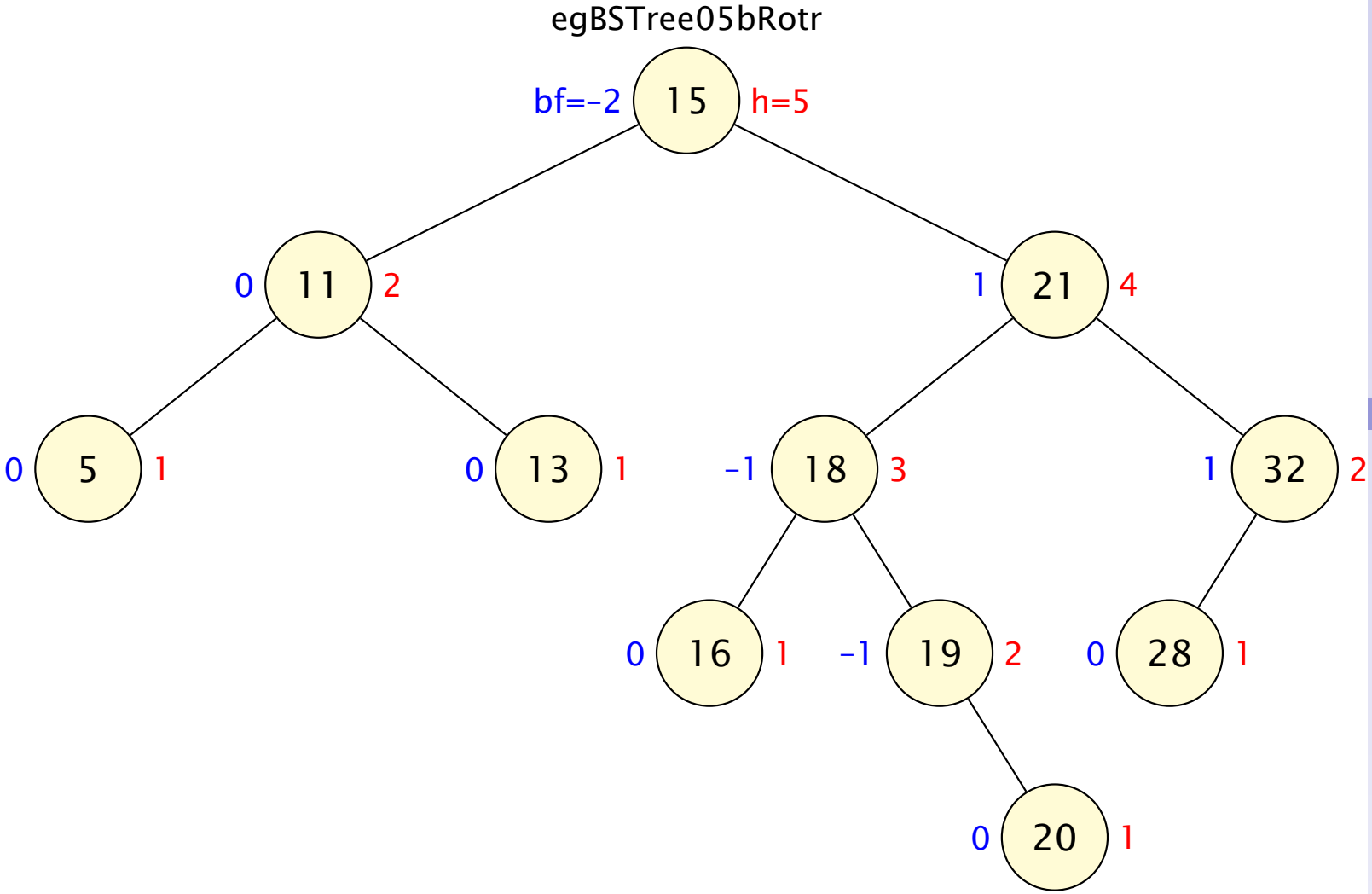
Answer 20 Add Item LR 2

- Here is the result of a right rotation around node with key 21



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LR Heavy

- This has just switched the balance factor of the root of the tree from 2 to -2 so we have to do something else.
- The *Eureka* step is realising that we can break up the problematic subtree under node 18 by doing a left rotation around node 15 — this produces the following tree.

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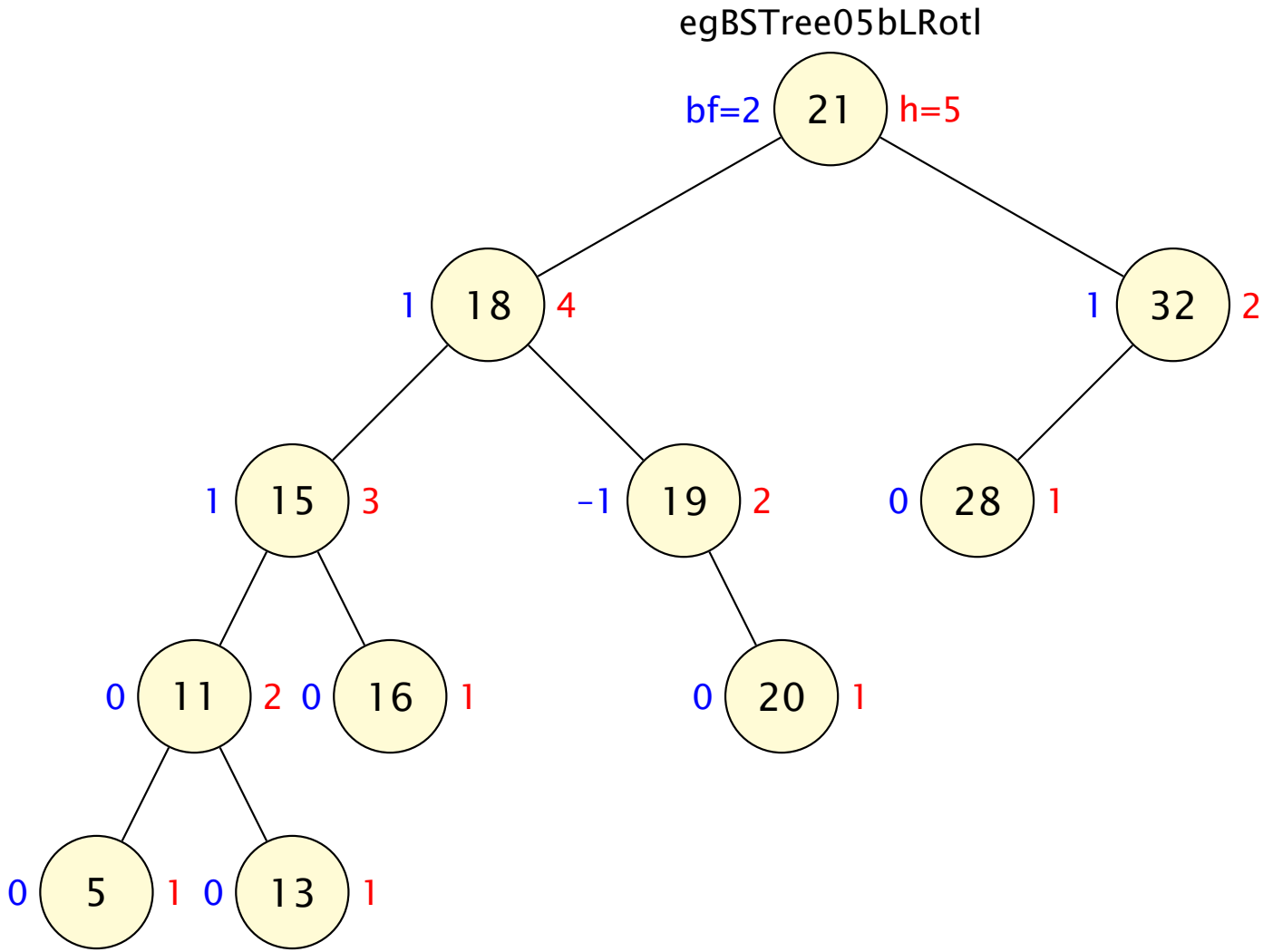
LR Heavy



- Notice that this has converted a tree which was LR heavy to one where it is LL heavy — so we can now use a right rotation on the tree rooted at 21 to get the following:

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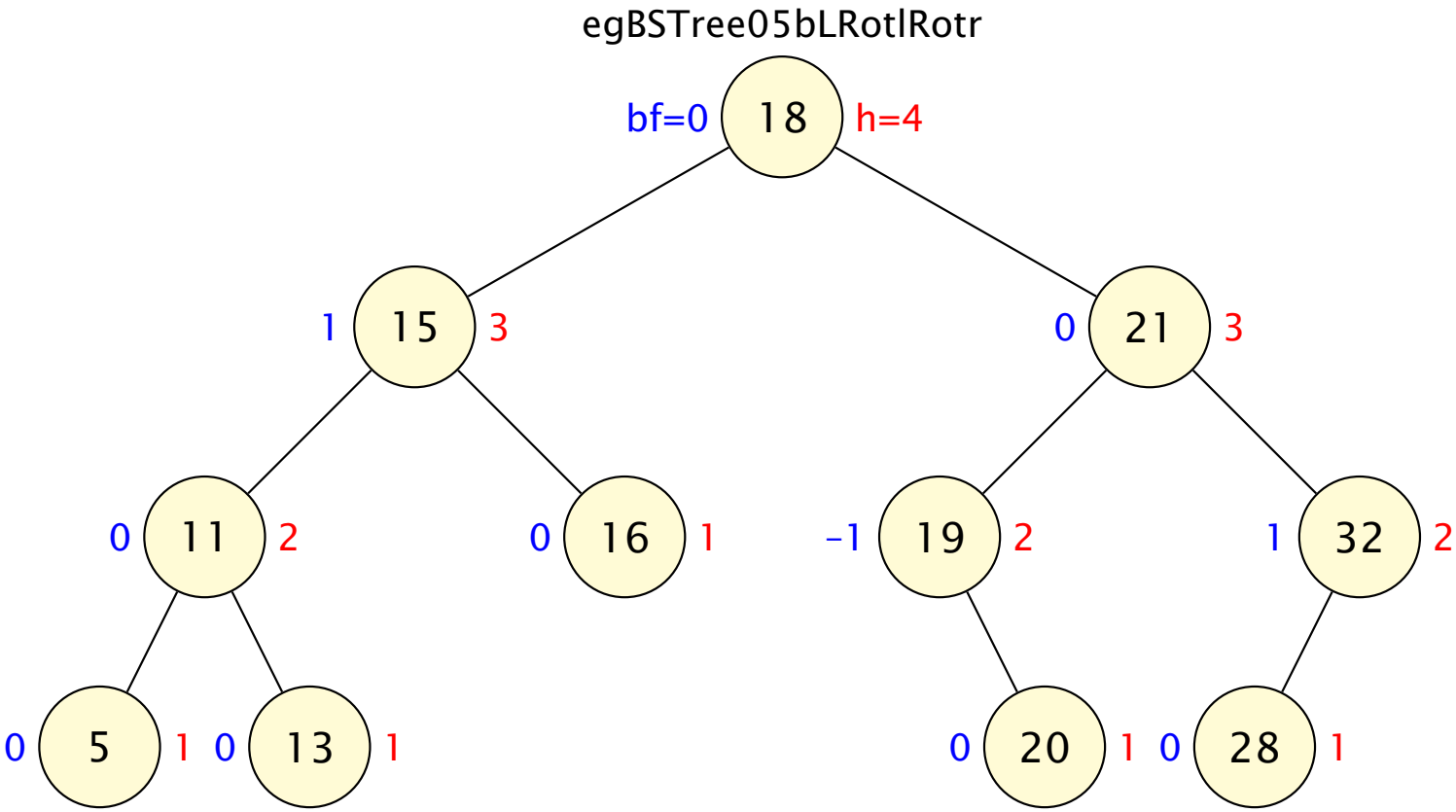
LR Heavy



- We now have a balanced tree — but were we just lucky or have we found a general rule ? Here are diagrams of the double rotation to show it works in general:

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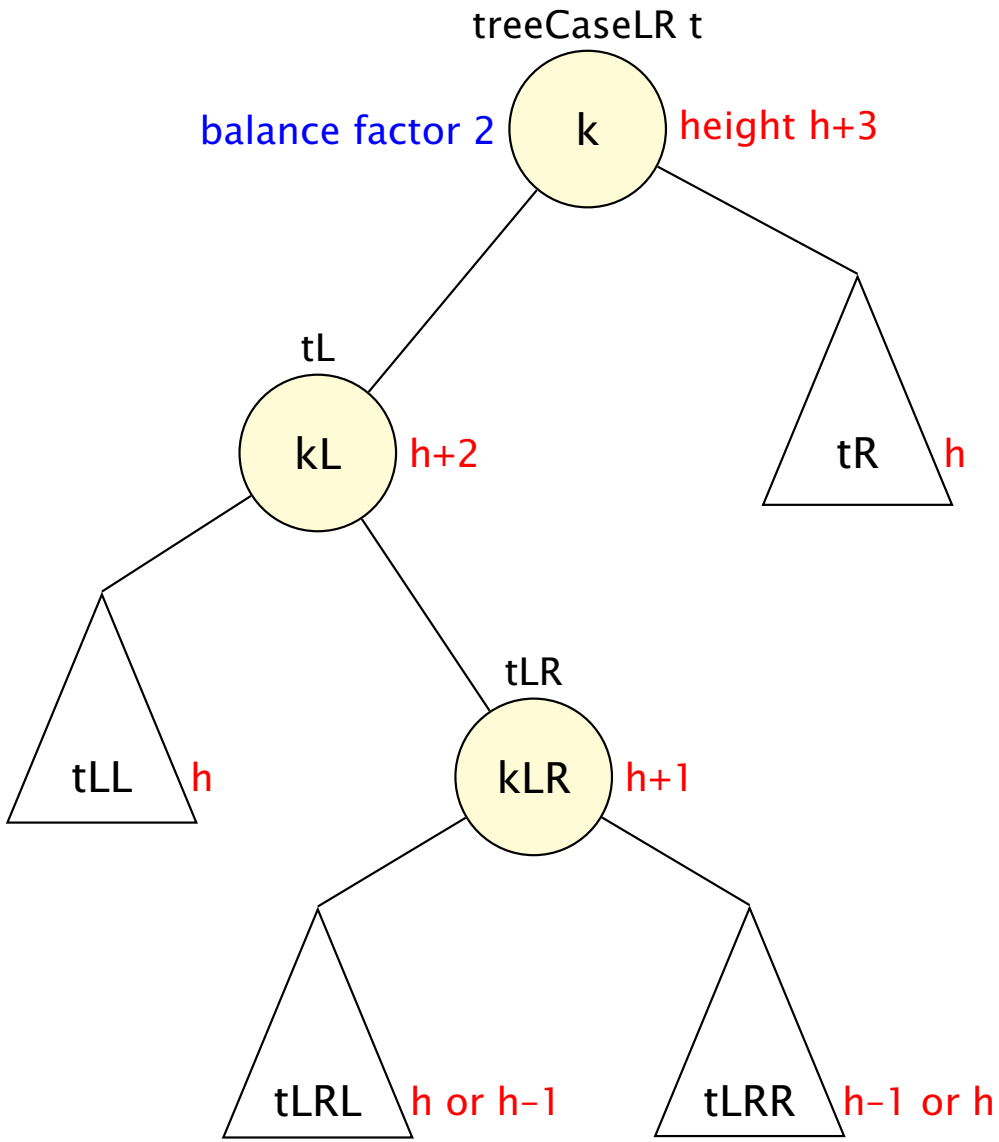
Case LR subsubtree heavy



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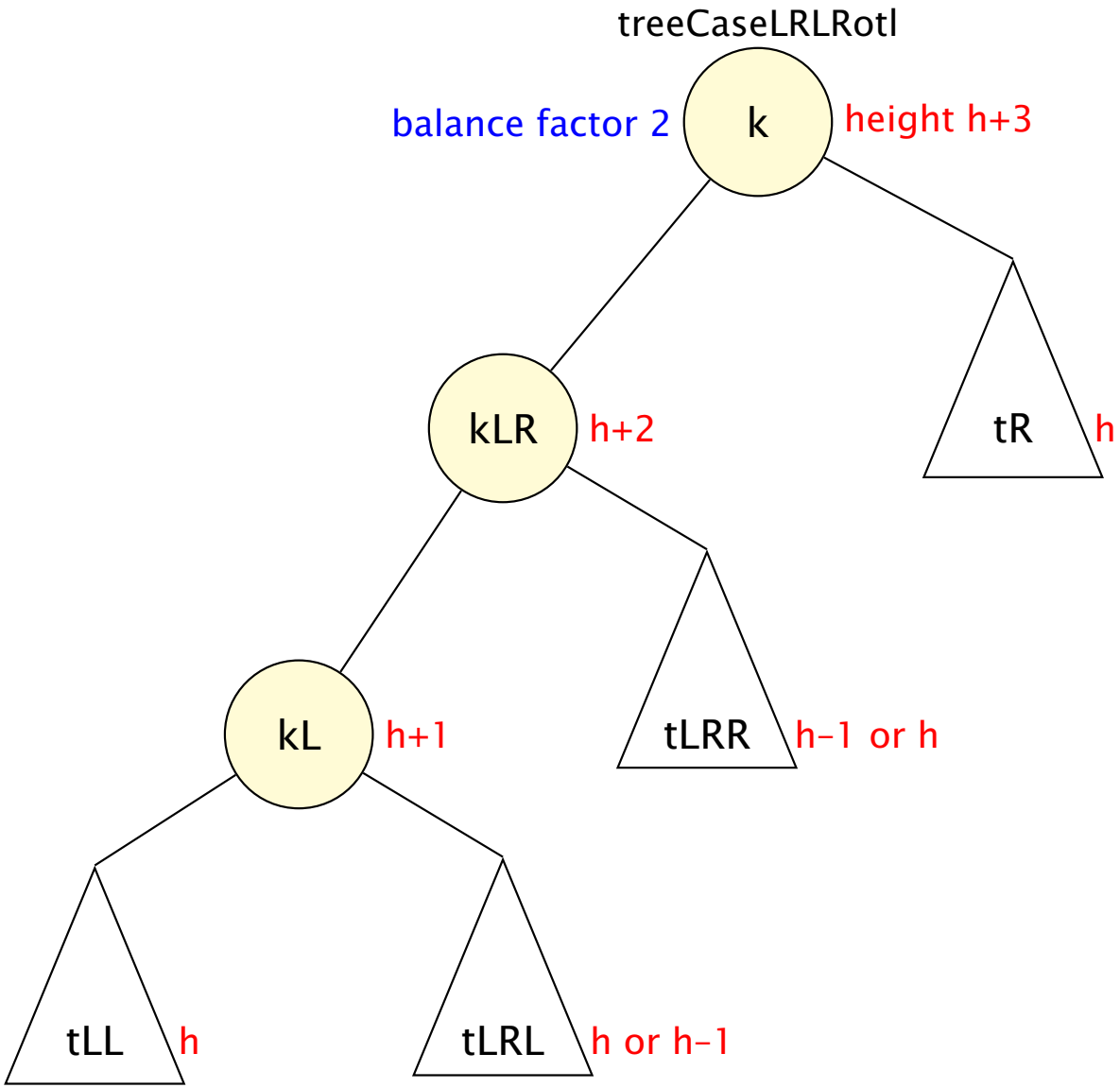
Case LR — Step (A) Rotate Left about kL



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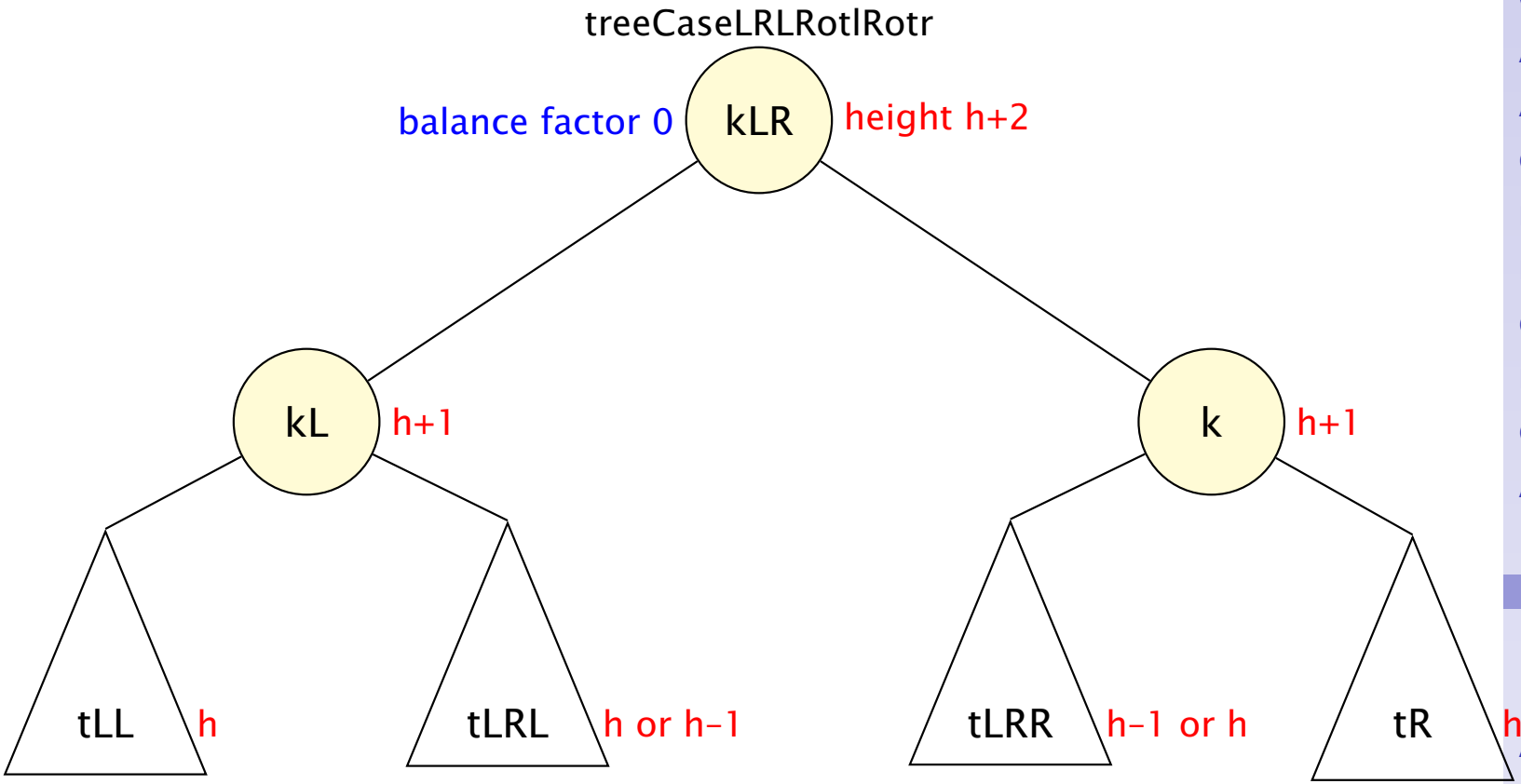
Case LR — Step (B) Rotate Right about k



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Activity 21 Case RL Heavy

- Draw the equivalent diagram for the final case where subsubtree `tRL` is heavy
- Note that this must be the mirror image of the `tLR` case

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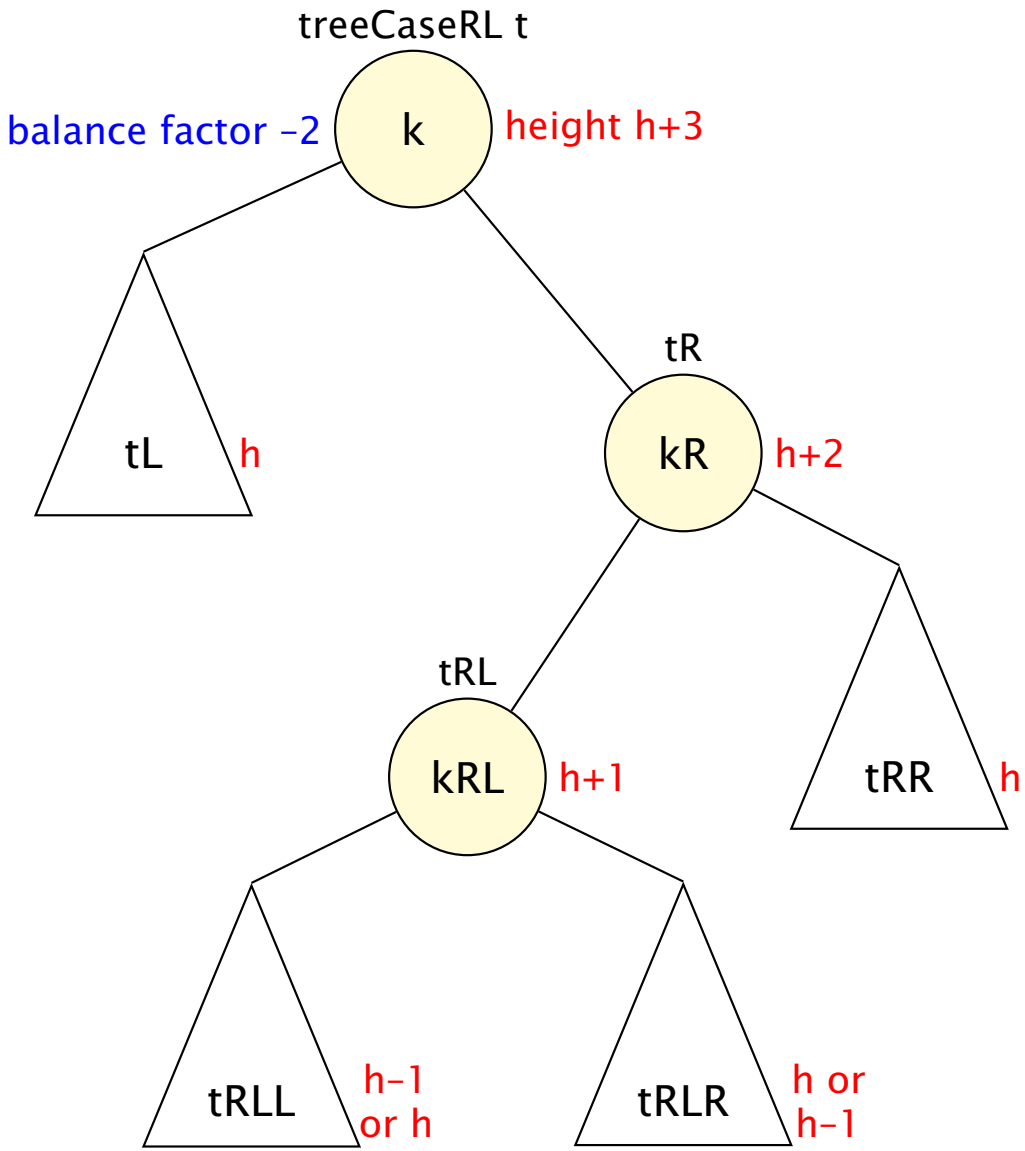
Answer 21 Case RL Heavy



- Answer 21 continued on next slide

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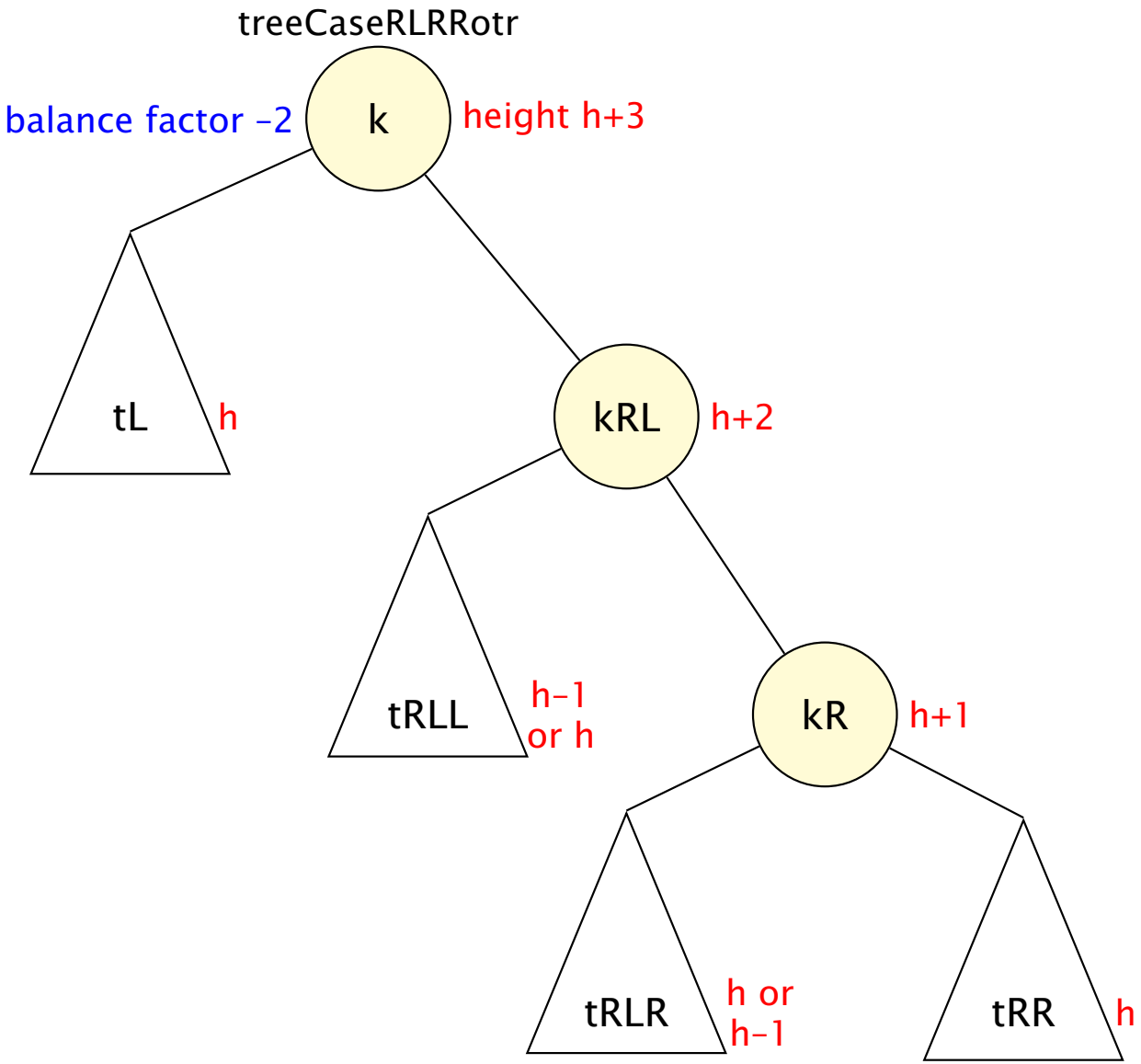
Answer 21 Case RL Heavy — Step (A) Rotate Right about kR



- Answer 21 continued on next slide

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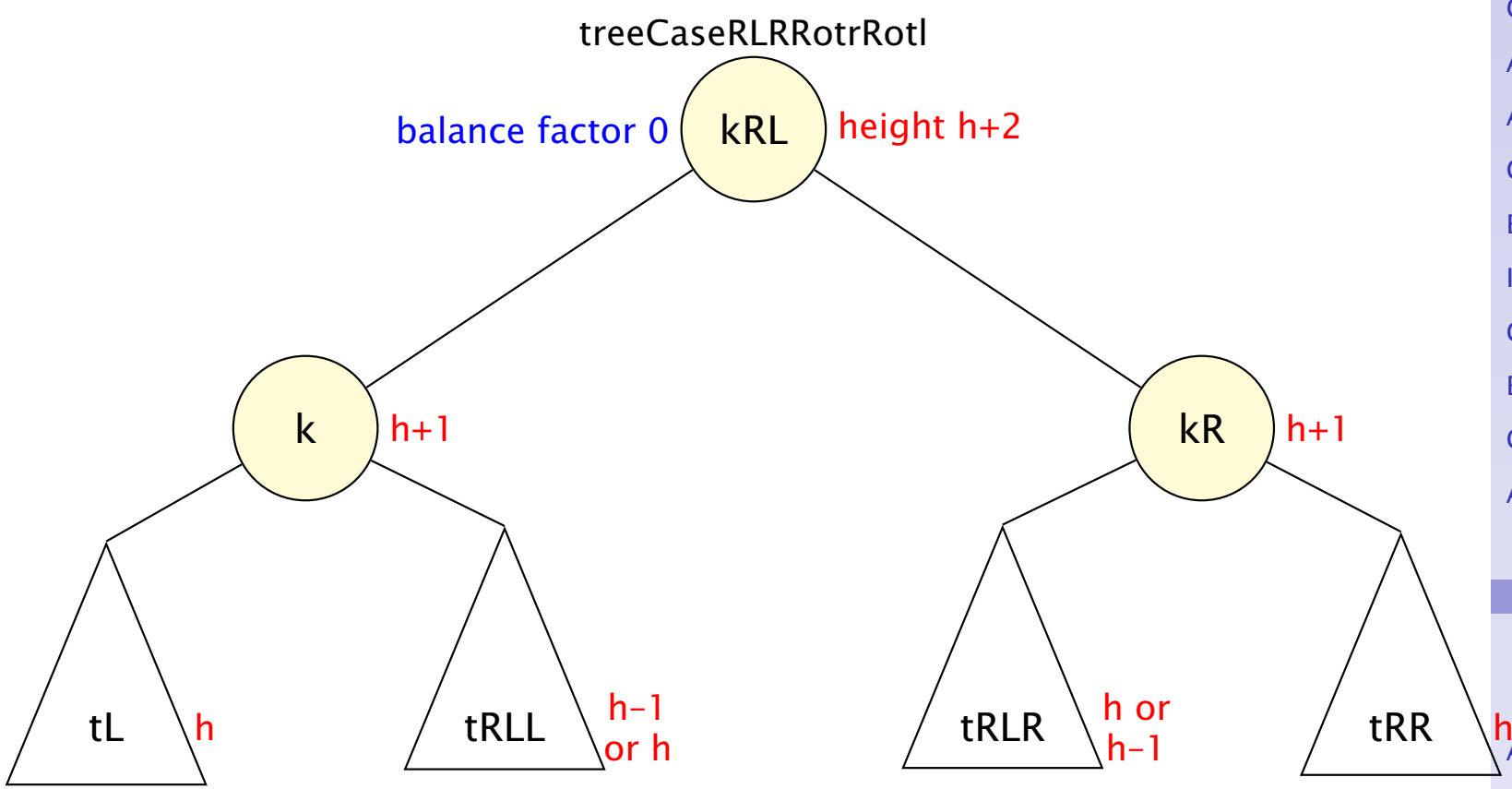
Answer 21 Case RL Heavy — Step (B) Rotate Left about k



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AVL Trees

The makeAVLTree Function

- The `makeAVLTree` takes an item, `x`, two subtrees, `leftT`, `rightT` and returns a new augmented binary tree
- It would be the same as `makeABTree` except it has to do the appropriate transformation if the new tree would be out of balance.
- We will only ever use `makeAVLTree` when inserting or deleting an item in a valid AVL tree
- So we know from our insight above that the heights of `leftT` and `rightT` can differ by at most 2 after insertion/deletion
- Hence we consider each case in turn using the transformations we have developed above.

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The makeAVLTree Function

Case 1 LL Heavy

```
(getHeightABT(leftT) - getHeightABT(rightT) = 2
 and balFactorABT(leftT) >= 0)
```

- Do a right rotation of the tree formed from `makeABTree(x, leftT, rightT)`

Case 2 LR Heavy

```
(getHeightABT(leftL) - getHeightABT(rightT) = 2
 and balFactorABT(leftT) == -1)
```

- Do a left rotation of `leftT`
- Do a right rotation of the tree formed from `makeABTree(x, rotl(leftT), rightT)`

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The makeAVLTree Function

Case 3 RL Heavy

```
(getHeightABT(leftL) - getHeightABT(rightT) = -2
 and balFactorABT(rightT) == 1)
```

- Do a right rotation of `rightT`
- Do a left rotation of the tree formed from `makeABTree(x, leftT, rotr(rightT))`

Case 4 RR Heavy

```
(getHeightABT(leftL) - getHeightABT(rightT) = -2
 and balFactorABT(rightT) <= 0)
```

- Do a left rotation of the tree formed from `makeABTree(x, leftT, rightT)`

Case 5 Otherwise

- Just use `makeABTree(x, leftT, rightT)`
- No transformations required

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makeAVLTree Function

Python

```
865 def makeAVLTree(x, leftT, rightT):
866   hL = getHeightABT(leftT)
867   hR = getHeightABT(rightT)
868   if (hR + 1 < hL) and (balFactorABT(leftT) >= 0):
869     return rotr(mkNodeABT(x, leftT, rightT))
870   elif (hR + 1 < hL):
871     return rotr(mkNodeABT(x, (rotl(leftT)),rightT))
872   elif (hL + 1 < hR) and (balFactorABT(rightT) > 0):
873     return rotl(mkNodeABT(x, leftT, rotr(rightT)))
874   elif (hL + 1 < hR):
875     return rotl(mkNodeABT(x, leftT, rightT))
876   else:
877     return mkNodeABT(x, leftT, rightT)
```


Binary Trees

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Conclusions

- This section has been quite long but most of the space has been occupied with diagrams
- Some *implementations* can look quite tricky since they may be trying to avoid recursion or manipulate the data structures
- We will discuss efficiency and recursion removal in a later section.
- Here are diagrams of the two rotate functions to emphasise that they are really quite simple.

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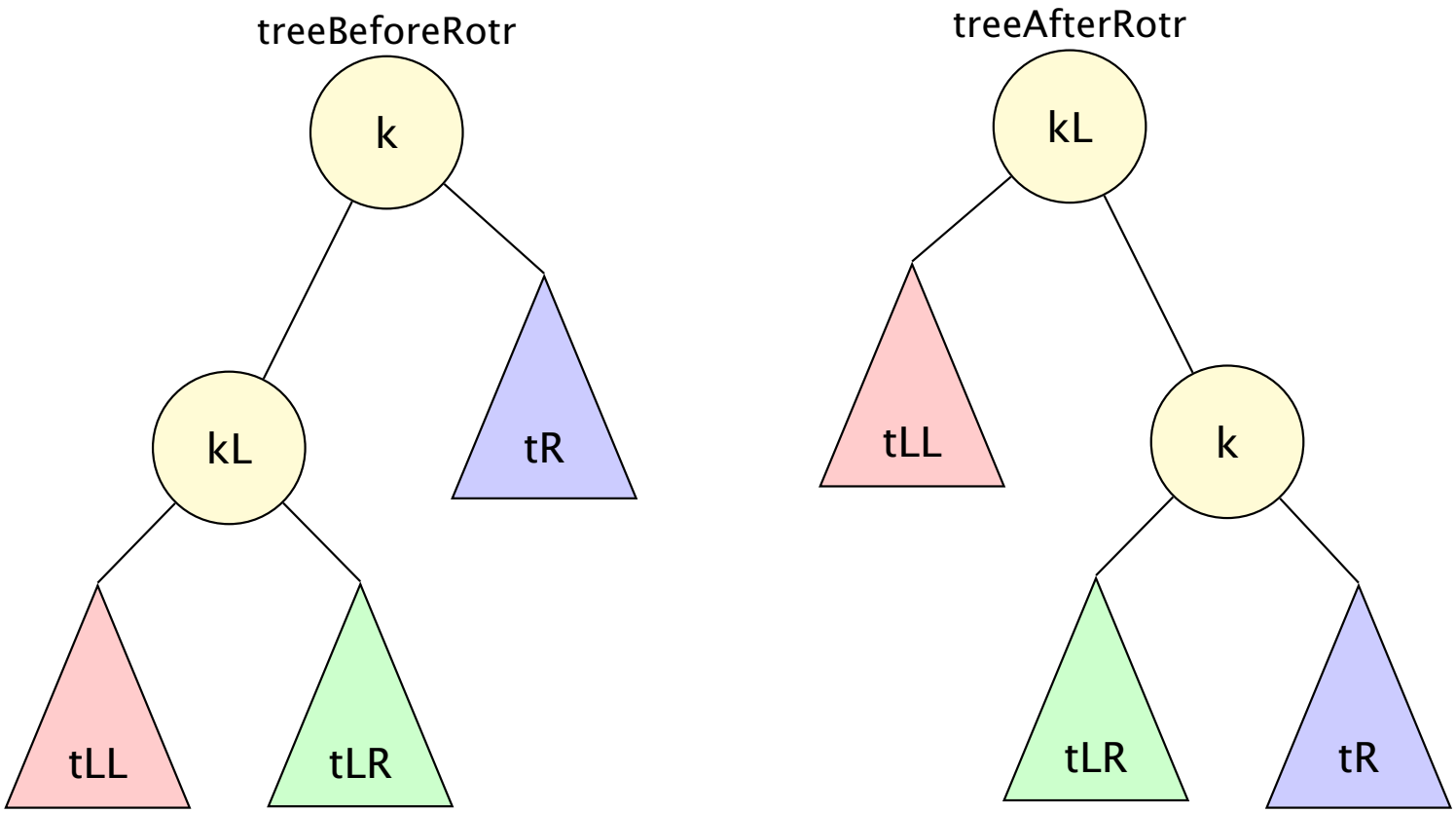
Rotate Right



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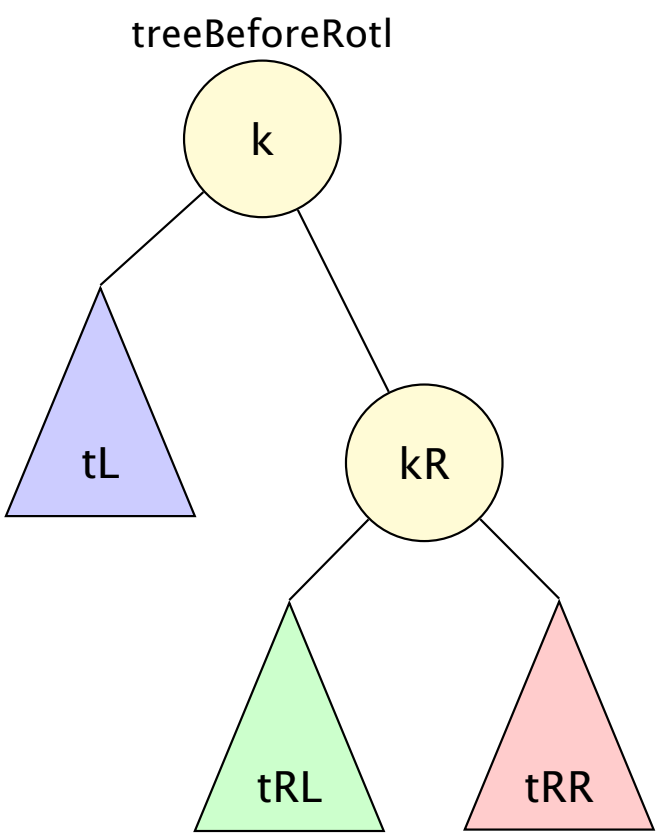
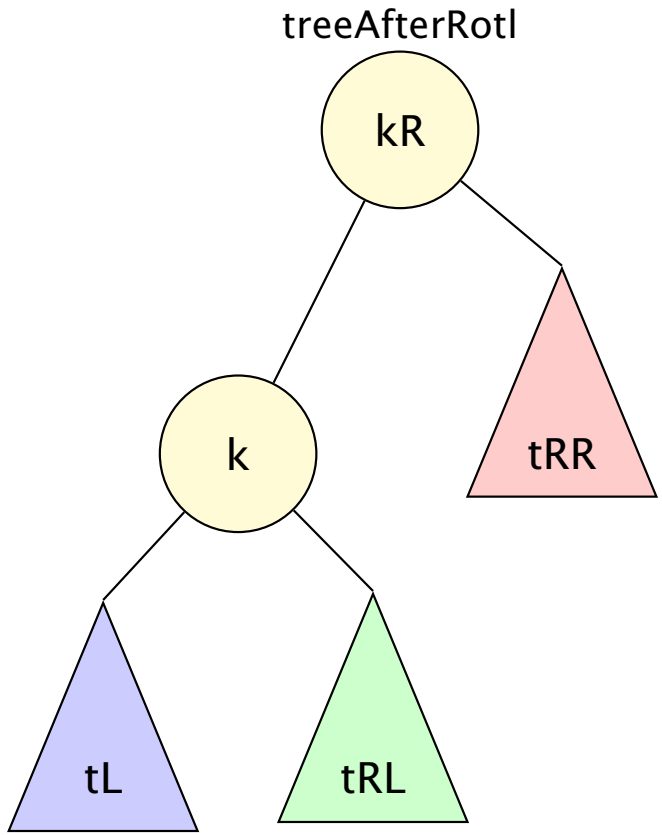
Rotate Left





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AVL Trees

Comparison with Storing Balance Factors

- Some texts implement AVL Trees by storing balance factors at the nodes rather than the heights (Miller and Ranum, 2011, Section 6.8.2, page 290)
- The Miller and Ranum explanation of updating the balance factors after a right or left rotation refer to diagrams similar to rotate right on slide 212 and rotate left on slide 213
- This note translates the Miller & Ranum notation to the notation used in these diagrams
- Both approaches have performance $O(\log n)$ but have differences in detail

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Comparison with Storing Balance Factors

Right Rotation (1)

- New and old balance factors of node k
- Using pseudo-code:

```
newBal(k) = height(tLR) - height(tR)
oldBal(k) = oldHeight(kL) - height(tR)
          = (1 + max(height(tLL), height(tLR)))
            - height(tR)

newBal(k) - oldBal(k)
  = (height(tLR) - height(tR))
    - ((1 + max(height(tLL), height(tLR)))
       - height(tR))
  = height(tLR)
    - 1 - max(height(tLL), height(tLR))
  = height(tLR)
    - 1 + min(-height(tLL), -height(tLR))
  = min(height(tLR) - height(tLL)
      , height(tLR) - height(tLR)) - 1
  = min(-oldBal(kL), 0) - 1
    since -max(a, b)    = min(-a,-b)
           min(a,b) + c = min(a+c, b+c)
```


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Comparison with Storing Balance Factors

Right Rotation (2)

- New and old balance factors of node kL

```
newBal(kL) = height(tLL) - newHeight(k)
           = height(tLL)
             - (1 + max(height(tLR), height(tR)))
oldBal(kL) = height(tLL) - height(tLR)

newBal(kL) - oldBal(kL)
  = (height(tLL)
    - (1 + max(height(tLR), height(tR))))
       - (height(tLL) - height(tLR))
  = height(tLR)
    - 1 - max(height(tLR), height(tR))
  = height(tLR)
    - 1 + min(-height(tLR), -height(tR))
  = min(height(tLR) - height(tLR)
      , height(tLR) - height(tR)) - 1
  = min(0, newBal(k)) - 1
```


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Comparison with Storing Balance Factors

Right Rotation (3)

- Right rotation: New and old balance factors of nodes k and kR

```
newBal(k)
  = oldBal(k) + min(-oldBal(kL), 0) - 1

newBal(kL)
  = oldBal(kL) + min(0, newBal(k)) - 1
```

- This fits with the right rotation diagrams annotated with heights and balance factors on slide 181,

```
oldBal(k)  = +2
oldBal(kL) = +1
newBal(k)  =  0 = +2 + min(-1, 0) - 1
newBal(kL) =  0 = +1 + min(0, 0) - 1
```


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Comparison with Storing Balance Factors

Left Rotation (1)

- New and old balance factors of node k
- Using pseudo-code:

```
newBal(k) = height(tL) - height(tRL)
oldBal(k) = height(tL) - oldHeight(kR)
          = height(tL)
            - (1 + max(height(tRL), height(tRR)))

newBal(k) - oldBal(k)
  = (height(tL) - height(tRL))
    - (height(tL)
       - (1 + max(height(tRL), height(tRR))))
  = 1 + max(height(tRL), height(tRR)) - height(tRL)
  = 1 + max(height(tRL) - height(tRL)
          , height(tRR) - height(tRL))
  = 1 + max(0, -oldBal(kR))
  = 1 - min(0, oldBal(kR))
    since max(-a, -b)  = -min(a,b)
          max(a,b) - c = max(a-c, b-c)
```


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Comparison with Storing Balance Factors

Left Rotation (2)

- New and old balance factors of node kR

```
newBal(kR) = newHeight(k) - height(tRR)
           = (1 + max(height(tL), height(tRL)))
             - height(tRR)
oldBal(kR) = height(tRL) - height(tRR)

newBal(kR) - oldBal(kR)
  = ((1 + max(height(tL), height(tRL)))
    - height(tRR))
       - (height(tRL) - height(tRR))
  = 1 + max(height(tL), height(tRL)) - height(tRL)
  = 1 + max(height(tL) - height(tRL)
          , height(tRL) - height(tRL))
  = 1 + max(newBal(k), 0)
```


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Comparison with Storing Balance Factors

Left Rotation (3)

- Left rotation: New and old balance factors of nodes k and kR

```
newBal(k)
  = oldBal(k) + 1 - min(0, oldBal(kR))

newBal(kR)
  = oldBal(kR) + 1 + max(newBal(k), 0)
```

- This fits with the left rotation diagrams annotated with heights and balance factors on slide 189,

```
oldBal(k)  = -2
oldBal(kR) = -1
newBal(k)  =  0 = -2 + 1 - min(0,-1)
newBal(kR) =  0 = -1 + 1 + max(0, 0)
```


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Insertion and Deletion

- The insertion and deletion functions are the same as for Binary Search Trees except we have to use makeAVLTree to make a tree unless we really know that the AVL property will be preserved.

```
881 def insertAVLT(x,t):
882   if isEmptyABT(t):
883     return mkNodeABT(x, mkEmptyABT(), mkEmptyABT())
884   else:
885     y = getDataABT(t)
886     leftT = getLeftABT(t)
887     rightT = getRightABT(t)
888     if x < y:
889       return makeAVLTree(y, insertAVLT(x, leftT), rightT)
890     elif x > y:
891       return makeAVLTree(y, leftT, insertAVLT(x, rightT))
892     else:
893       return t
```

```
895 def insertListAVLT(xs,t):
896   if xs == []:
897     return t
898   else:
899     return insertListAVLT(xs[1:], (insertAVLT(xs[0],t)))
```


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Insertion and Deletion

```
901 def deleteAVLT(x,t):
902   if isEmptyABT(t):
903     return mkEmptyABT()
904   else:
905     y = getDataABT(t)
906     leftT = getLeftABT(t)
907     rightT = getRightABT(t)
908     if x < y:
909       return makeAVLTree(y, deleteAVLT(x, leftT), rightT)
910     elif x > y:
911       return makeAVLTree(y, leftT, deleteAVLT(x, rightT))
912     else:
913       return joinAVLT(leftT, rightT)
```


Binary Trees

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Insertion and Deletion

```
917 def joinAVLT(leftT, rightT):
918   if isEmptyABT(rightT):
919     return leftT
920   else:
921     (y,t) = splitAVLT(rightT)
922     return makeAVLTree(y, leftT, t)
```

```
926 def splitAVLT(t):
927   if isEmptyABT(t):
928     raise RuntimeError("splitAVLT␣applied␣to␣EmptyABT()")
929   else:
930     x = getDataABT(t)
931     t1 = getLeftABT(t)
932     t2 = getRightABT(t)
933     if isEmptyABT(t1):
934       return (x,t2)
935     else:
936       (y,t3) = splitAVLT(t1)
937       return (y, makeAVLTree(x, t3, t2))
```


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Activity 22 Insert Lists and Delete Items

- Draw the AVL Trees resulting from inserting the following lists of items into an empty tree one by one in order given — do the insertions by hand following the AVL insertion algorithm — you can use the Python code to check your answers
  1. [1,2,3,4,5,6,7,8,9,10]
  2. [10,9,8,7,6,5,4,3,2,1]
  3. [68,88,61,89,94,50,4,76,66,82,99]
- For each of the previous trees, show the result when the fourth item inserted is deleted
- The `insertAVLT` function is defined at line 881, slide 221 (Python), the `deleteAVLT` function is defined at line 901, slide 222 (Python),
- `insertListAVLT` is defined at line 895, slide 221 (Python),

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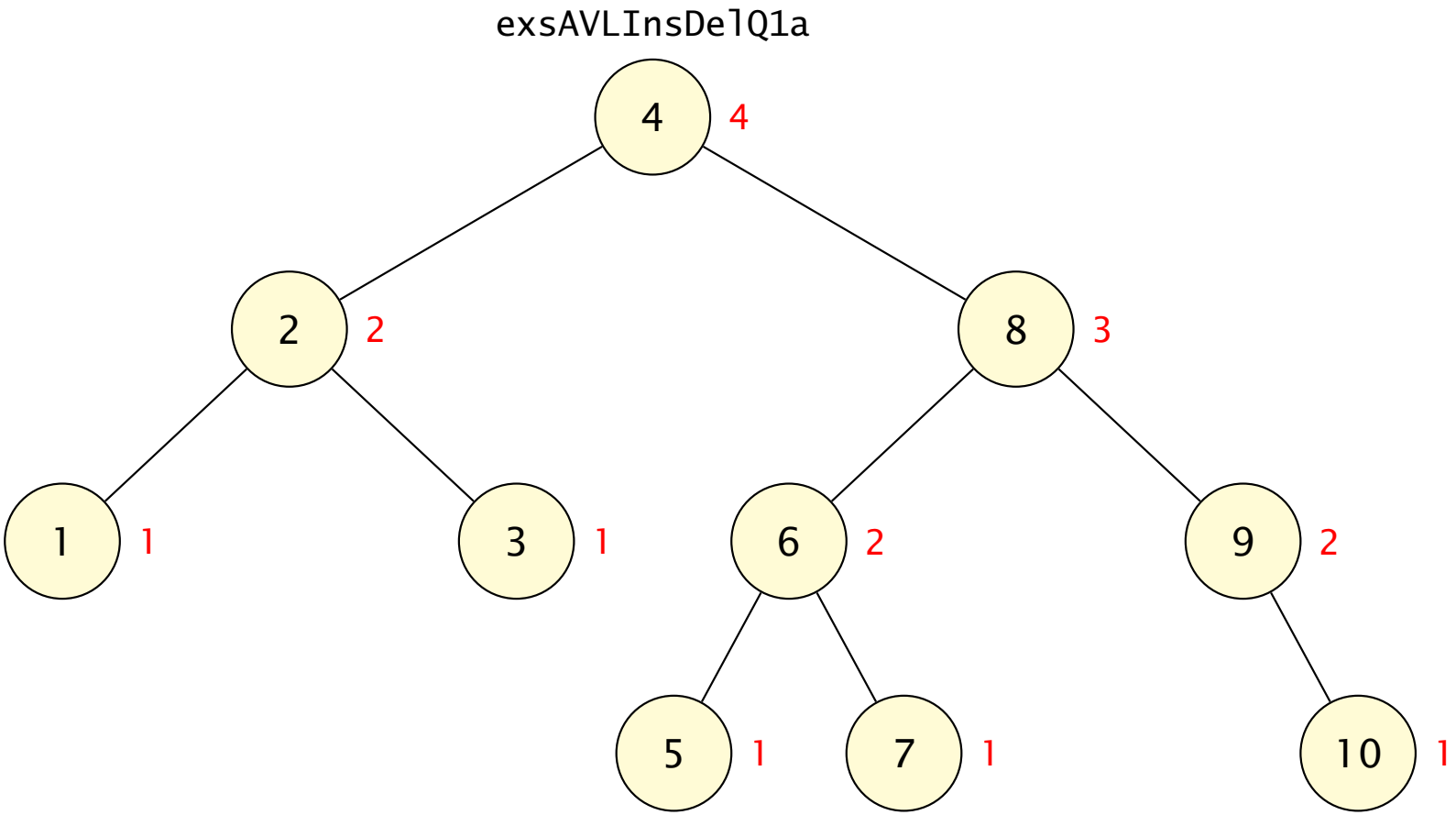
Answer 22 Insert Lists and Delete Items

```
listQ1a = [1,2,3,4,5,6,7,8,9,10]
exsAVLInsDelQ1a = insertListAVLT(listQ1a, EmptyABT())
```



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Answer 22 Insert Lists and Delete Items

- Here is the Python representation of the resulting AVL tree

```
exsAVLInsDelQ1aAns \
 = (NodeABT(4, 4,
      NodeABT(2, 2,
        NodeABT(1, 1, EmptyABT(), EmptyABT()),
        NodeABT(1, 3, EmptyABT(), EmptyABT())),
      NodeABT(3, 8,
        NodeABT(2, 6,
          NodeABT(1, 5, EmptyABT(), EmptyABT()),
          NodeABT(1, 7, EmptyABT(), EmptyABT())),
        NodeABT(2, 9,
          EmptyABT(),
          NodeABT(1, 10, EmptyABT(), EmptyABT())))))
```

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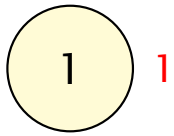
Answer 22 Insert Lists and Delete Items

- Here are the insertions done one by one with separate diagrams

```
exsAVLInsDelQ1a01 \
 = insertListAVLT(listQ1a[:1], EmptyABT())
```

exsAVLInsDelQ1a01



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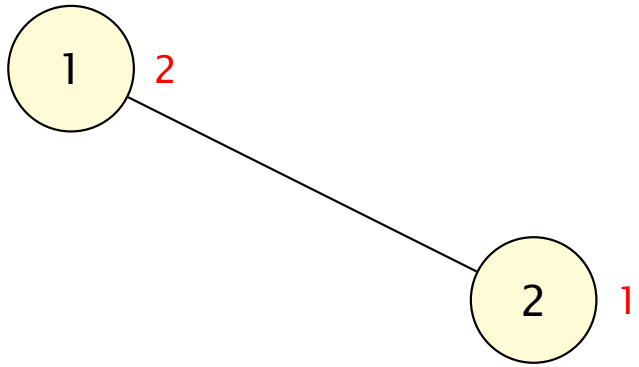
```
exsAVLInsDelQ1a02 \
 = insertListAVLT(listQ1a[:2], EmptyABT())
```

exsAVLInsDelQ1a02



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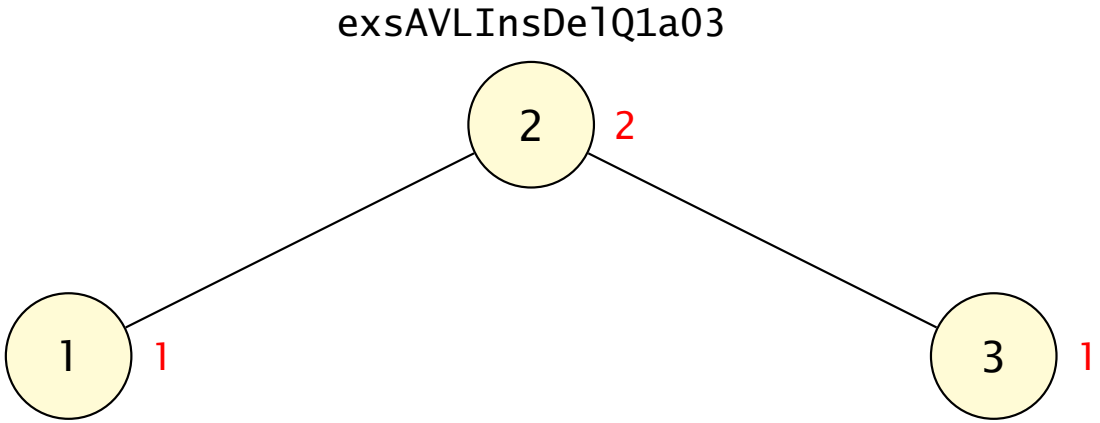
```
exsAVLInsDelQ1a03 \
 = insertListAVLT(listQ1a[:3], EmptyABT())
```

Left rotation about 1



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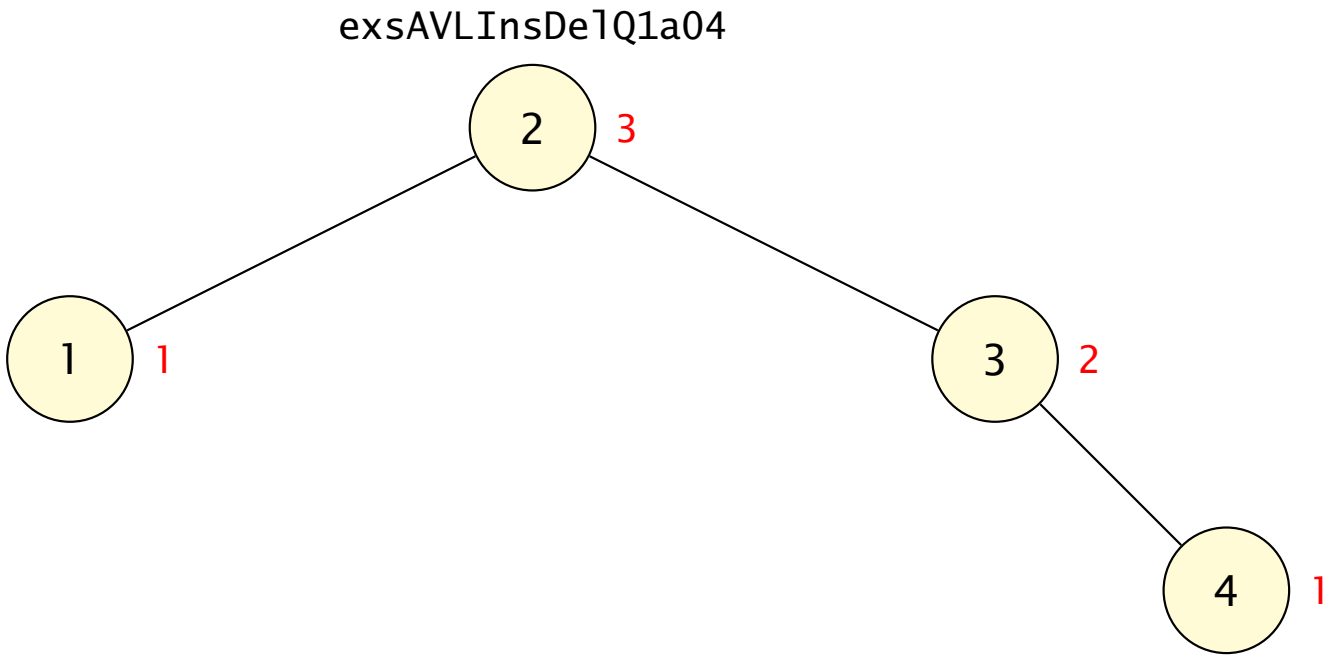
```
exsAVLInsDelQ1a04 \
 = insertListAVLT(listQ1a[:4], EmptyABT())
```



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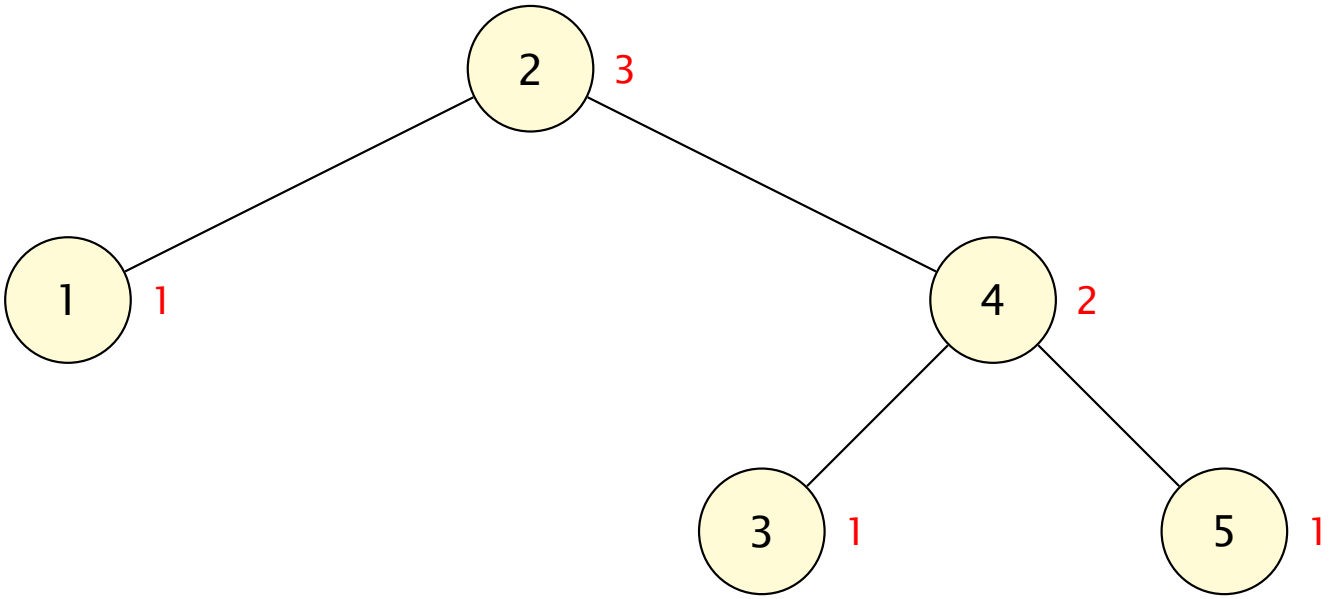
```
exsAVLInsDelQ1a05 \
 = insertListAVLT(listQ1a[:5], EmptyABT())
```

Left rotation about 3

exsAVLInsDelQ1a05



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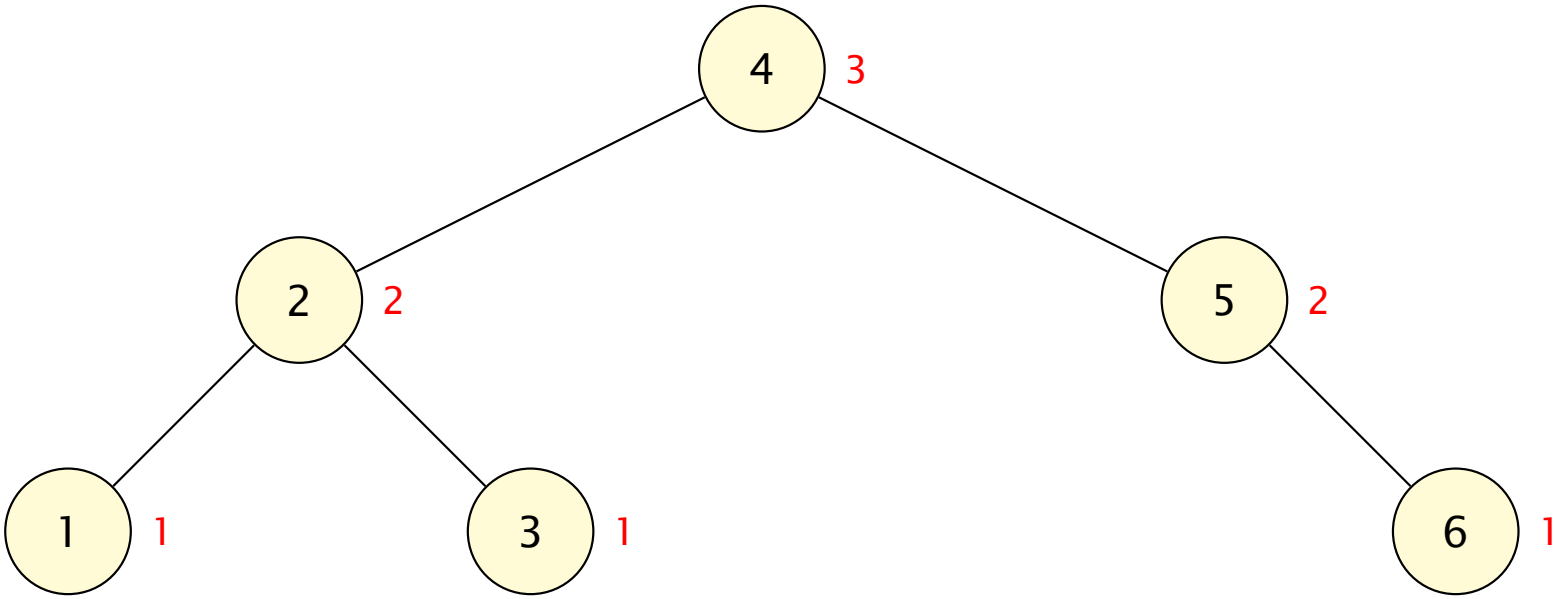
```
exsAVLInsDelQ1a06 \
 = insertListAVLT(listQ1a[:6], EmptyABT())
```

Left rotation about 2

exsAVLInsDelQ1a06



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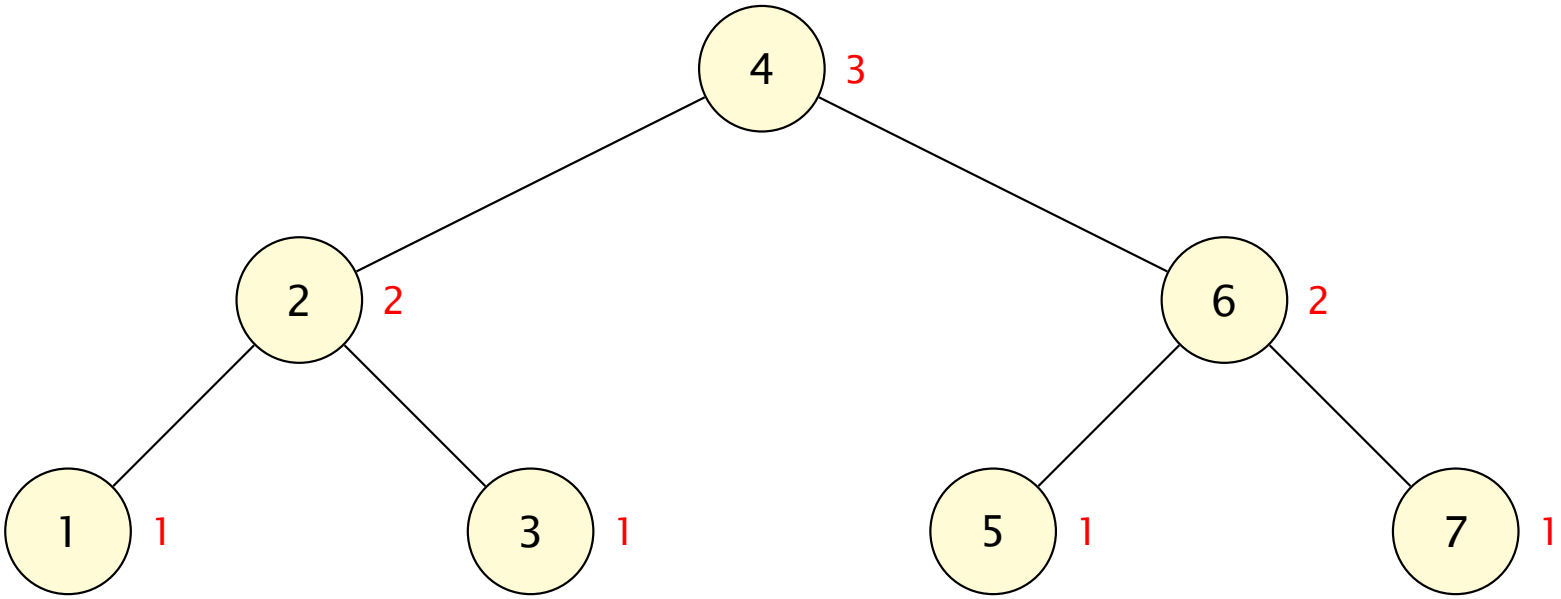
```
exsAVLInsDelQ1a07 \
 = insertListAVLT(listQ1a[:7], EmptyABT())
```

Left rotation about 5

exsAVLInsDelQ1a07



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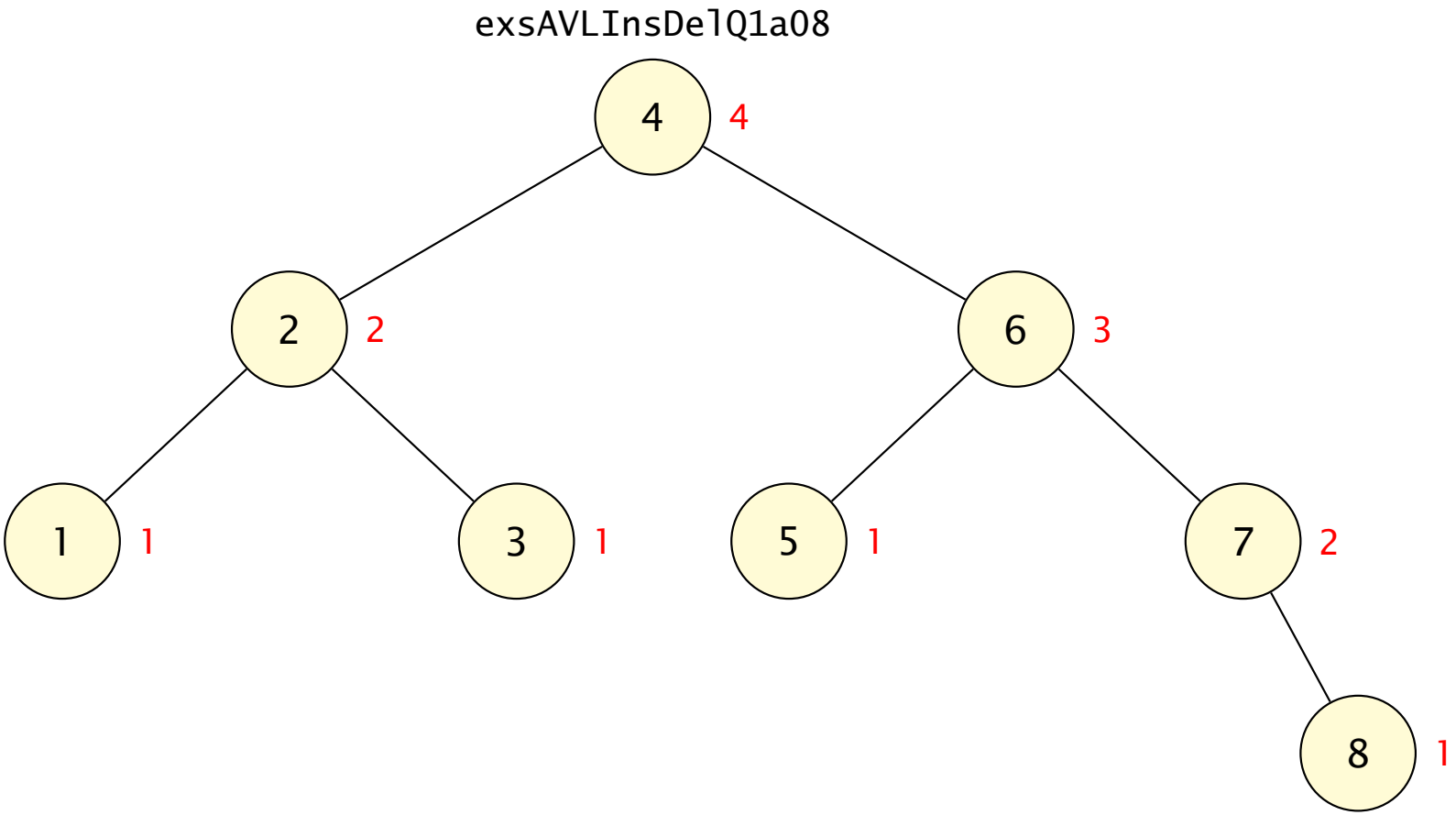
```
exsAVLInsDelQ1a08 \
 = insertListAVLT(listQ1a[:8], EmptyABT())
```



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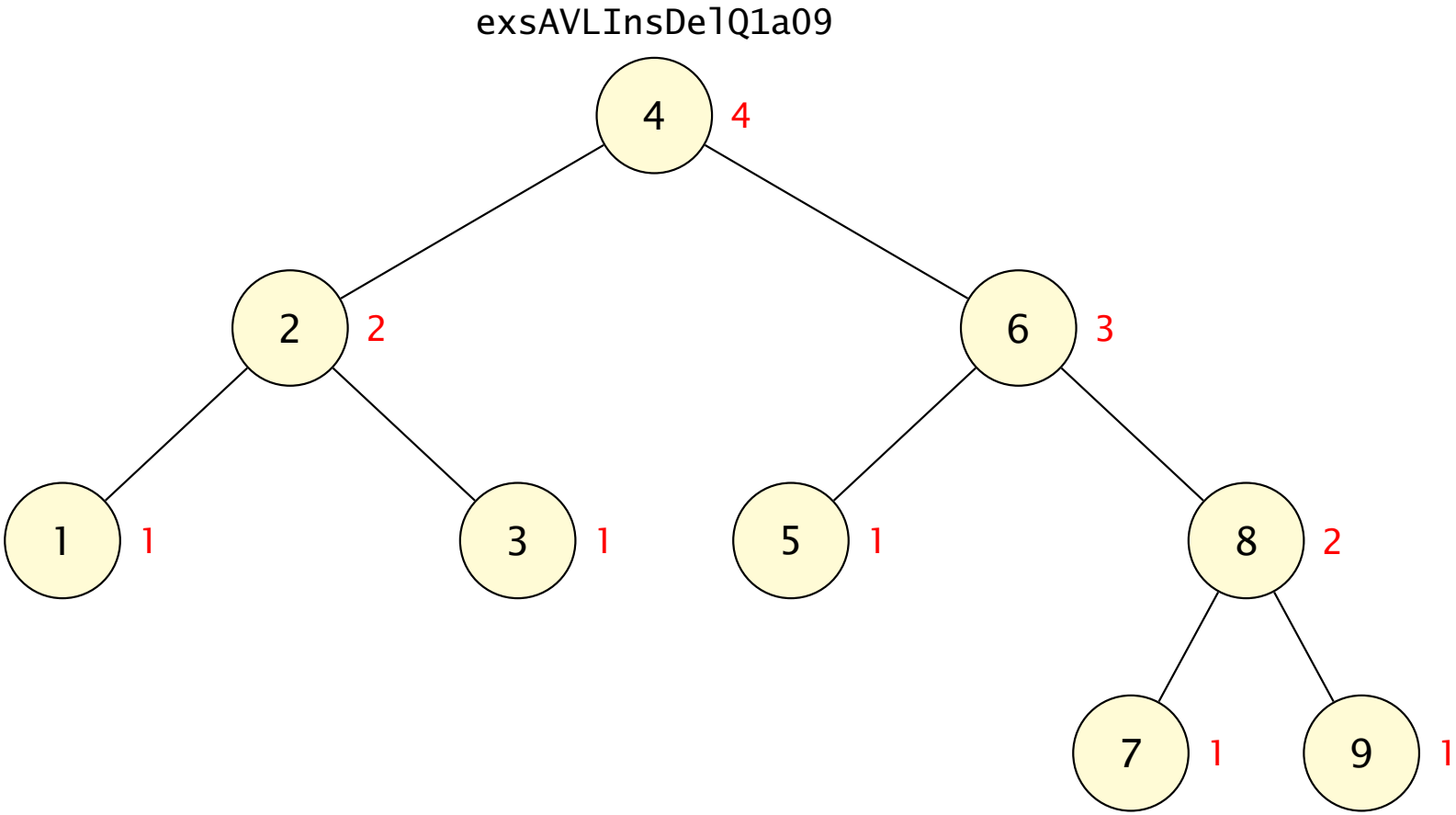
```
exsAVLInsDelQ1a09 \
 = insertListAVLT(listQ1a[:9], EmptyABT())
```

Left rotation about 7



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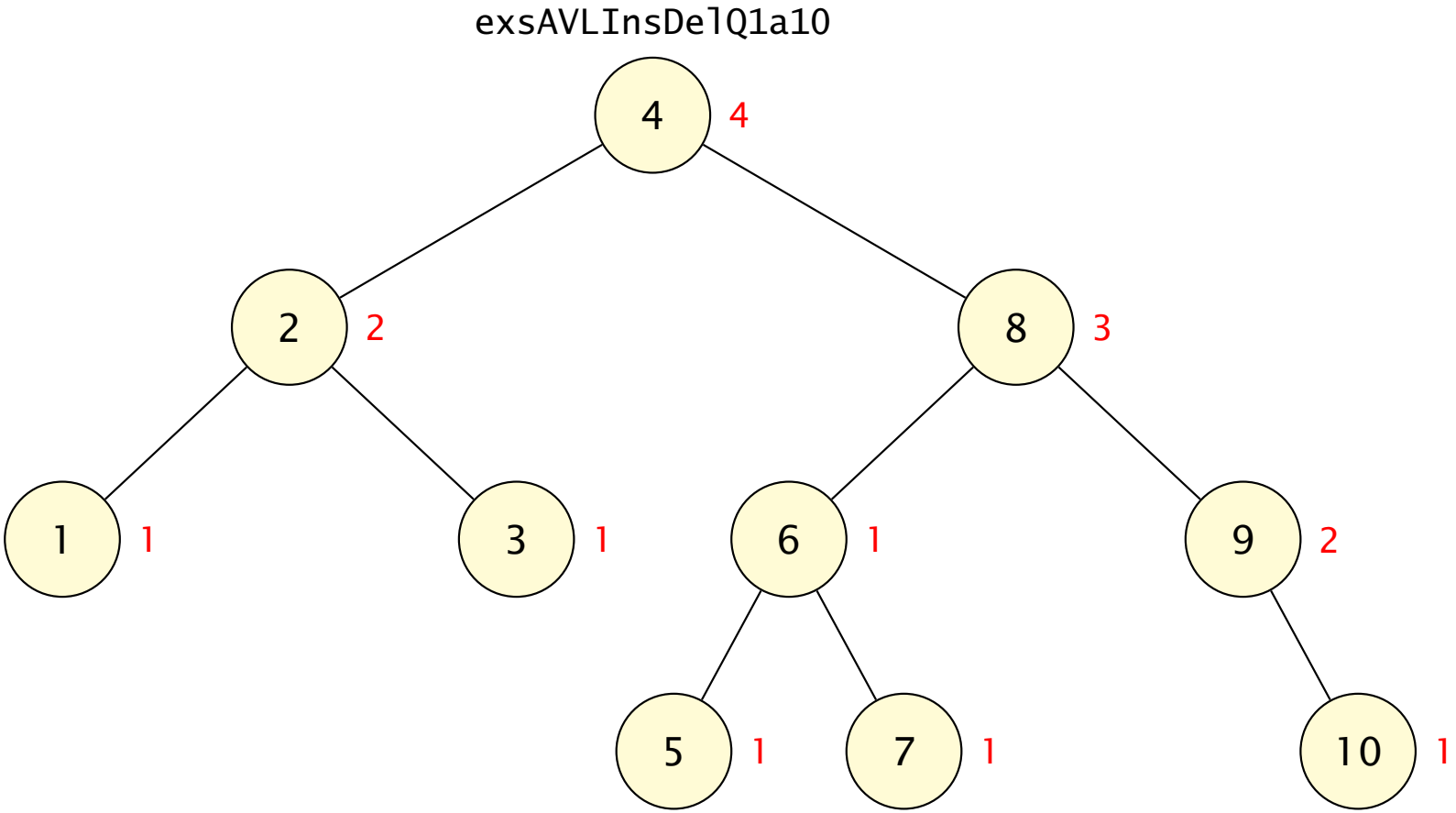
```
exsAVLInsDelQ1a10 \
 = insertListAVLT(listQ1a[:10], EmptyABT())
```

Left rotation around 6



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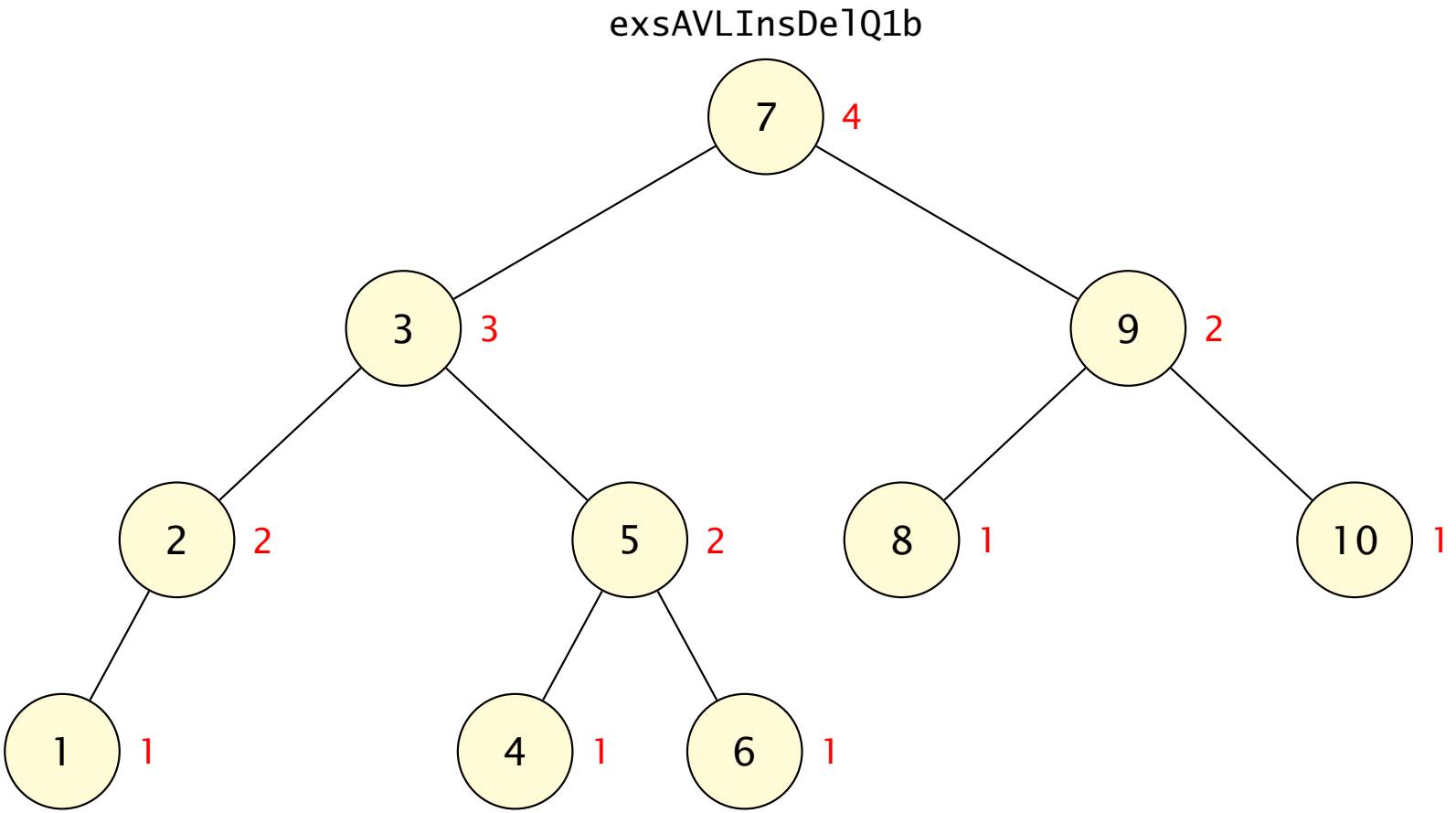
```
listQ1b = [10,9,8,7,6,5,4,3,2,1]
exsAVLInsDelQ1b = insertListAVLT(listQ1b, EmptyABT())
```



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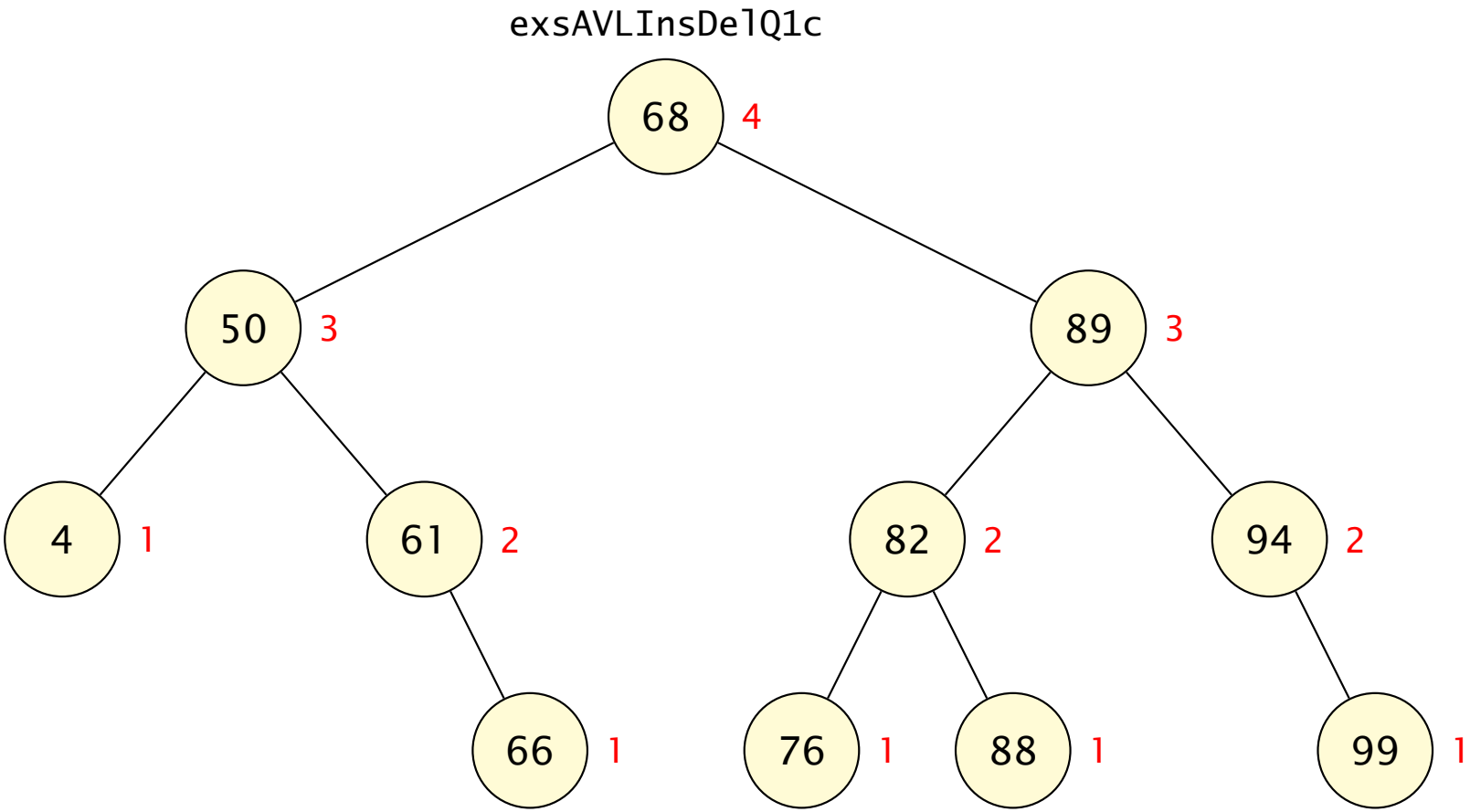
Answer 22 Insert Lists and Delete Items

```
listQ1c = [68,88,61,89,94,50,4,76,66,82,99]
exsAVLInsDelQ1b = insertListAVLT(listQ1c, EmptyABT())
```



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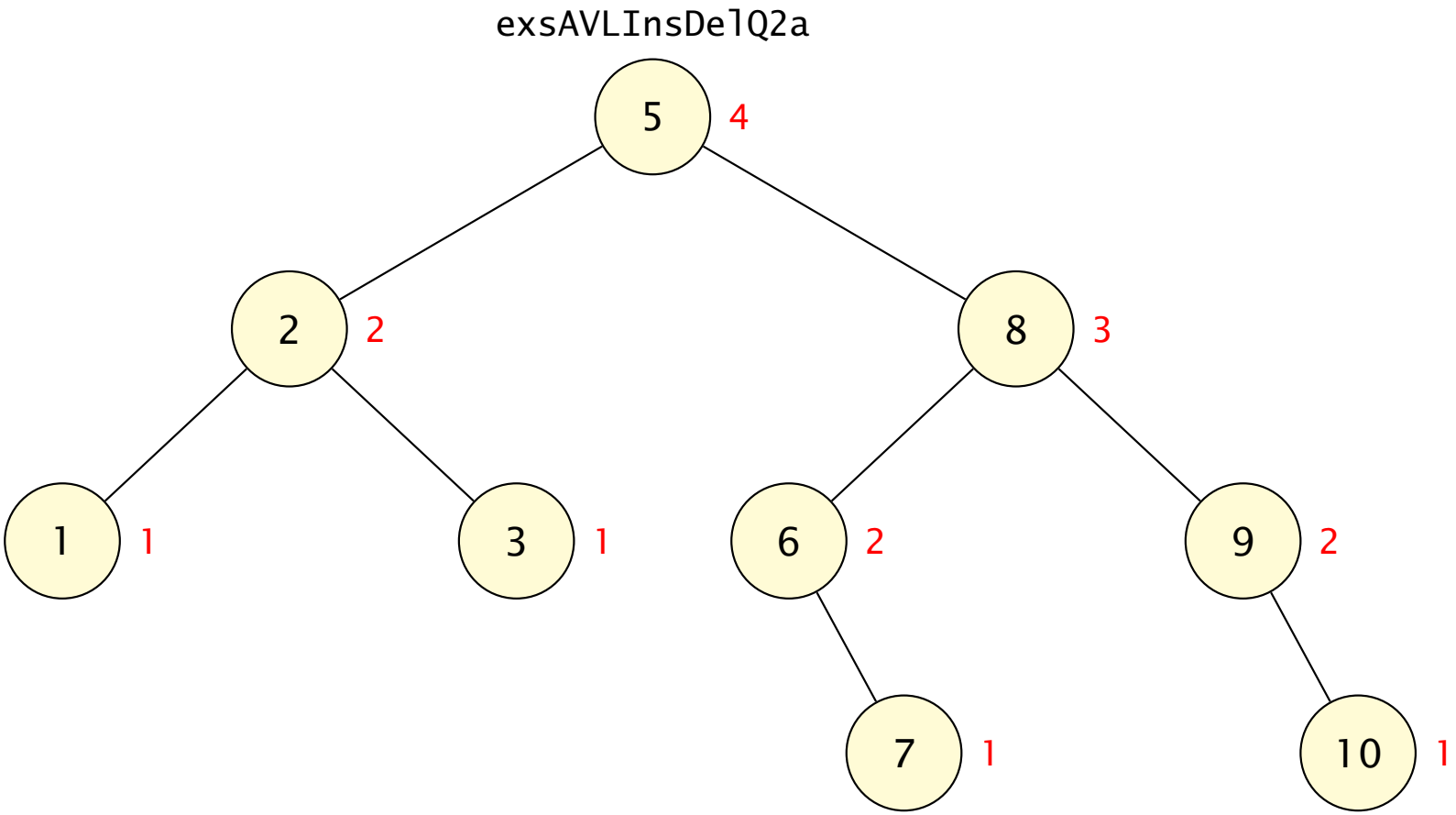
AVL Trees

Answer 22 Q2(a) Delete 4th Item

```
listQ1a = [1,2,3,4,5,6,7,8,9,10]
exsAVLInsDelQ2a \
 = deleteAVLT(listQ1a[3], exsAVLInsDelQ1a)
```



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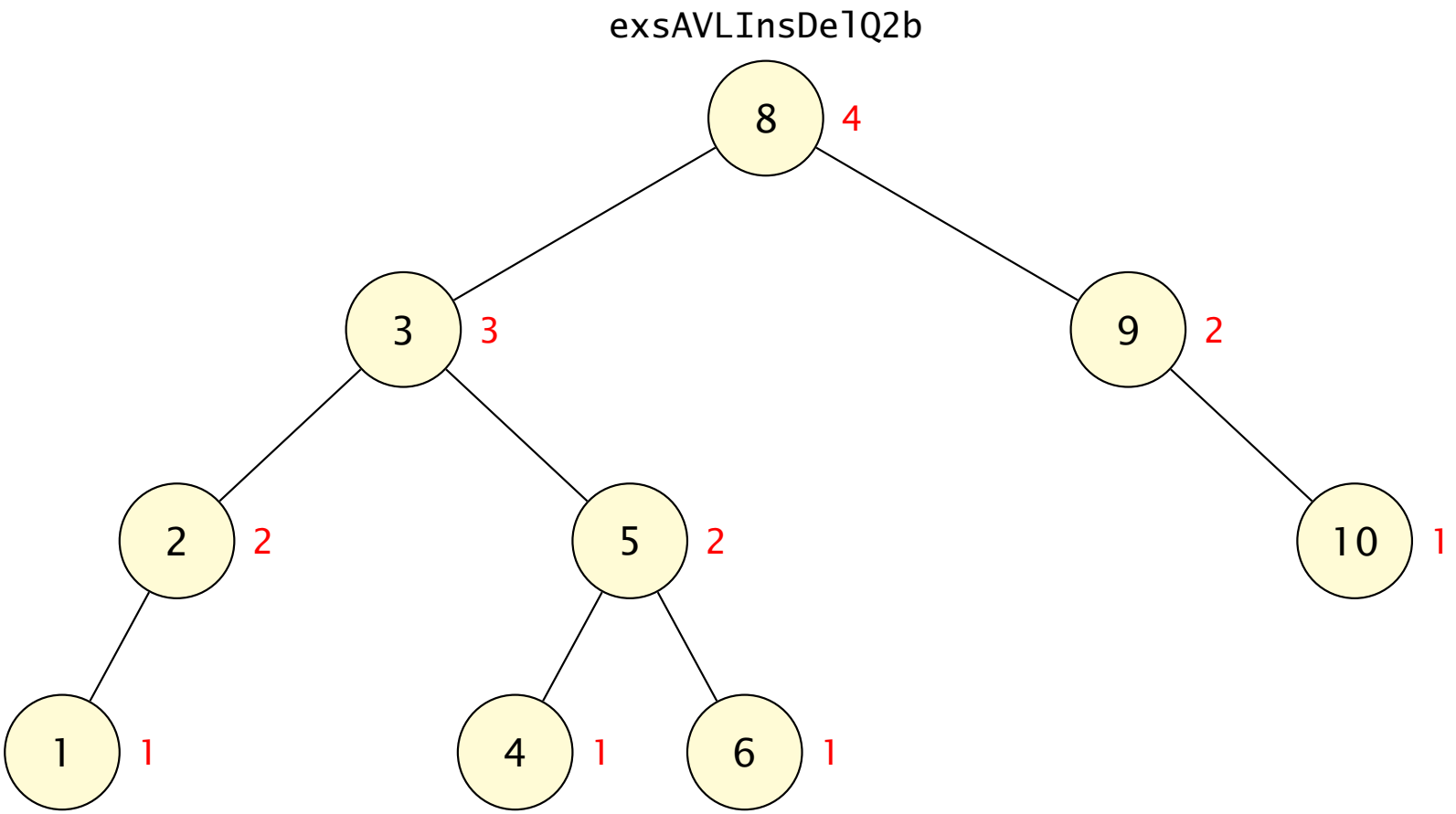
AVL Trees

Answer 22 Q2(b) Delete 4th Item

```
listQ1b = [10,9,8,7,6,5,4,3,2,1]
exsAVLInsDelQ2b \
 = deleteAVLT(listQ1b[3], exsAVLInsDelQ1b)
```



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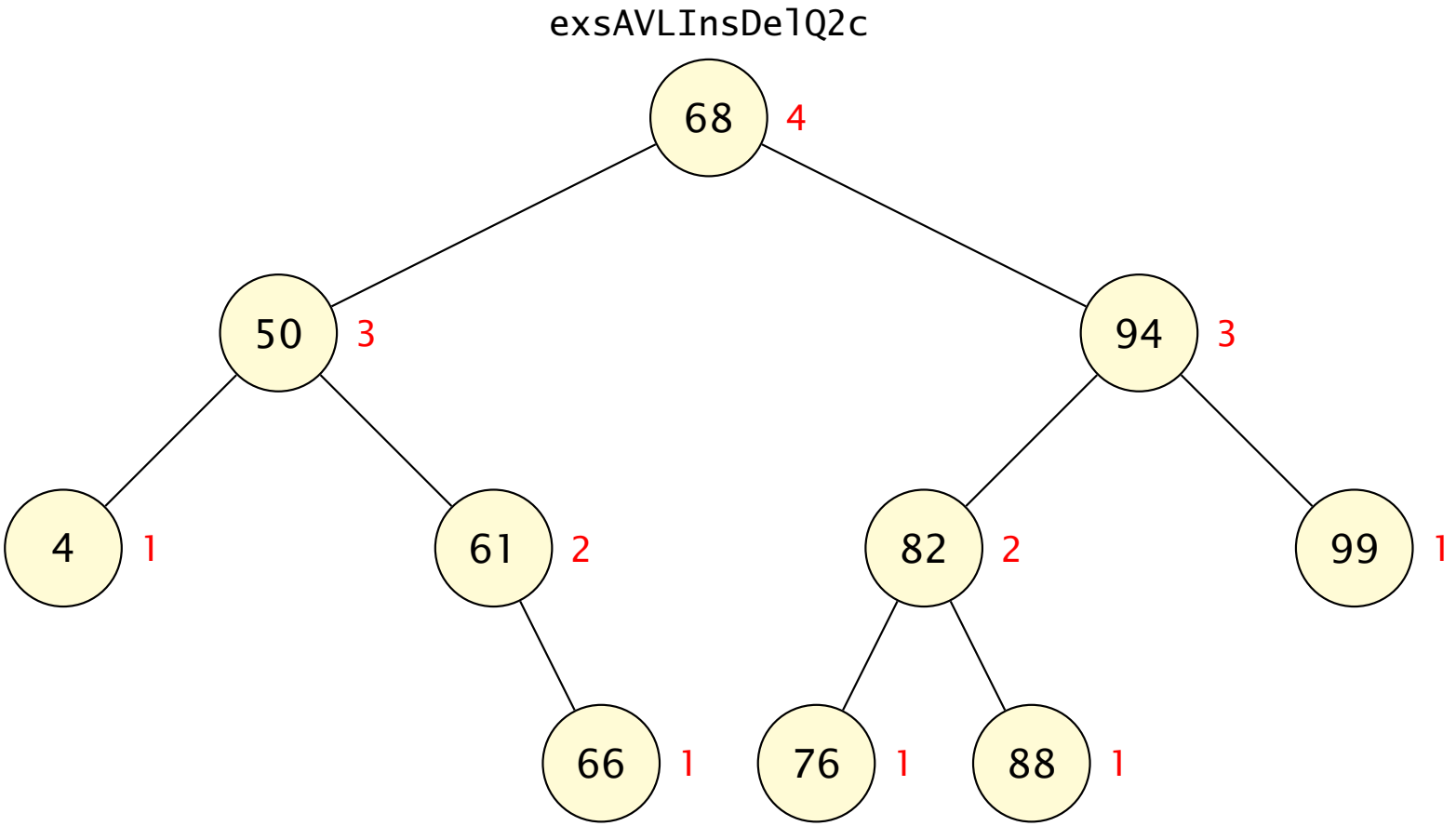
AVL Trees

Answer 22 Insert Lists and Delete Items

```
listQ1c = [68,88,61,89,94,50,4,76,66,82,99]
exsAVLInsDelQ2c \
 = deleteAVLT(listQ1c[3], exsAVLInsDelQ1c)
```



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AVL Tree Delete Insertion

Activity 23 Deleting Inserted List

- Using `listQ1a` show that deleting the elements of the list from the tree one by one in reverse order does not result in the reverse sequence of AVL trees

```
listQ1a = [1,2,3,4,5,6,7,8,9,10]
exsAVLInsDelQ1a = insertListAVLT(listQ1a, EmptyABT())
```

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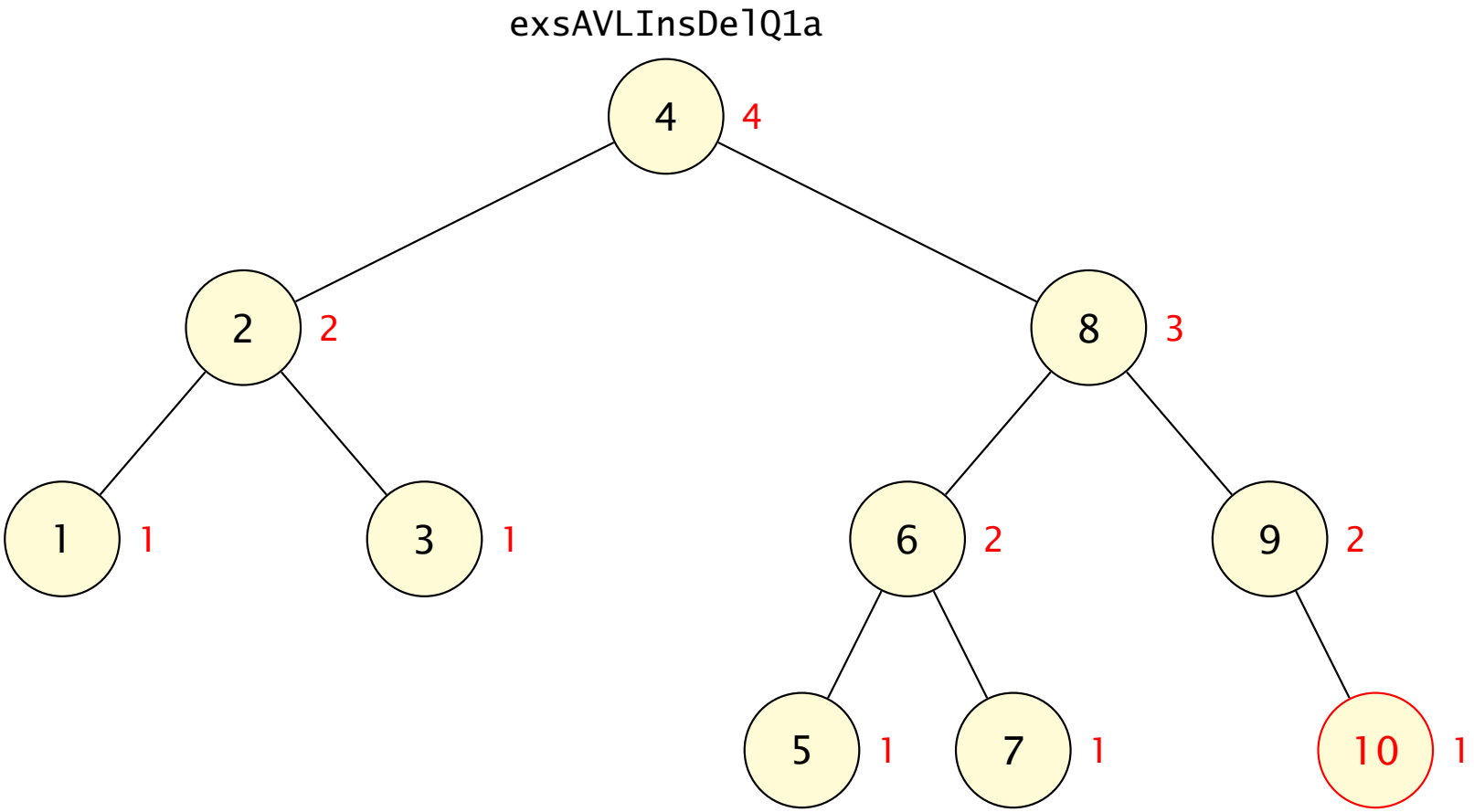
AVL Tree Delete Insertion

Answer 23 Deleting Inserted List

```
listQ1a = [1,2,3,4,5,6,7,8,9,10]
exsAVLInsDelQ1a = insertListAVLT(listQ1a, EmptyABT())
# delete listQ1a[-1]
```



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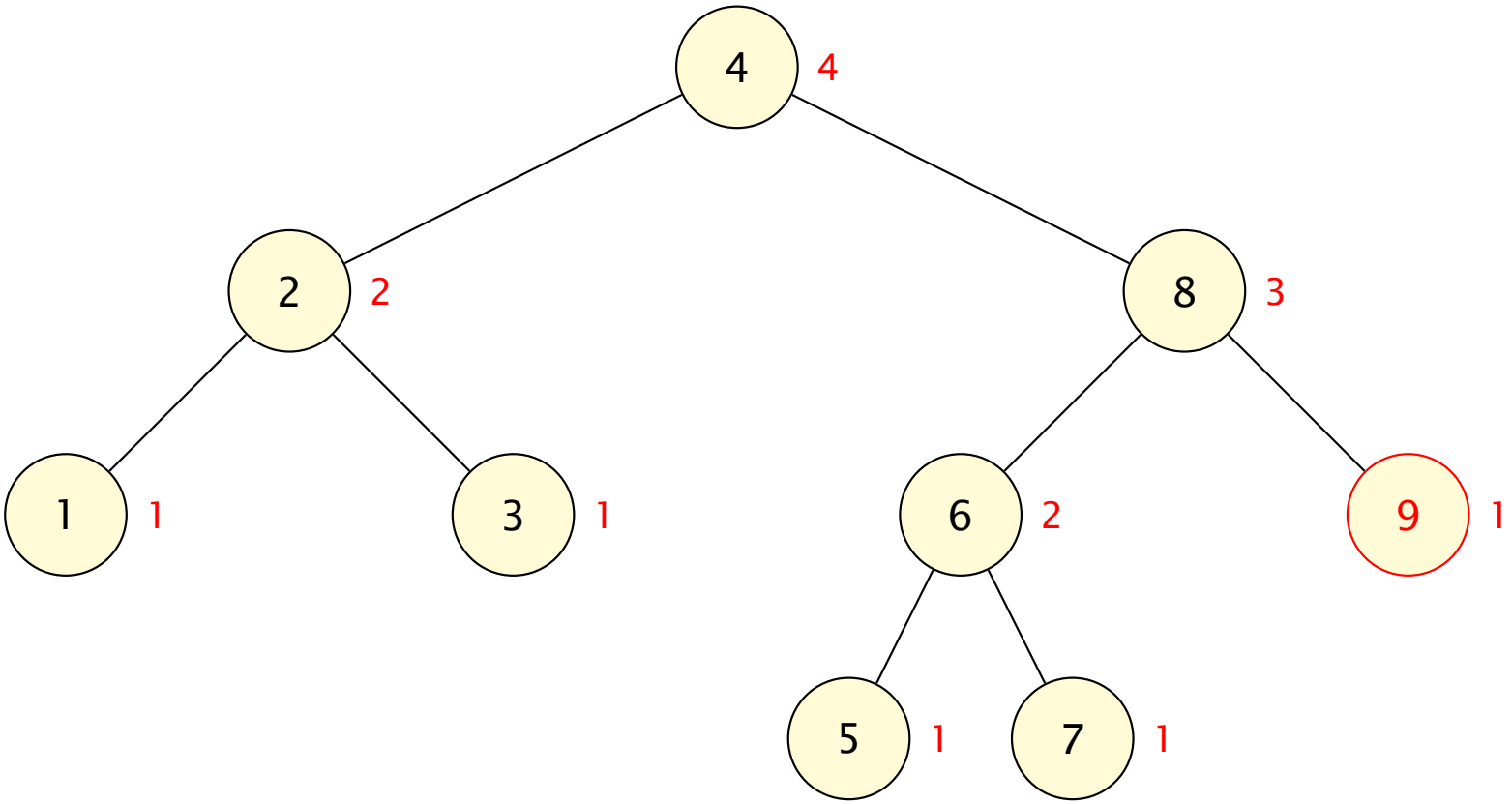
AVL Tree Delete Insertion

Answer 23 Deleting Inserted List

```
exsAVLDelListQ1a01 \
 = deleteAVLT(listQ1a[-1], exsAVLInsDelQ1a)
# delete listQ1a[-2]
```

exsAVLDelListQ1a01



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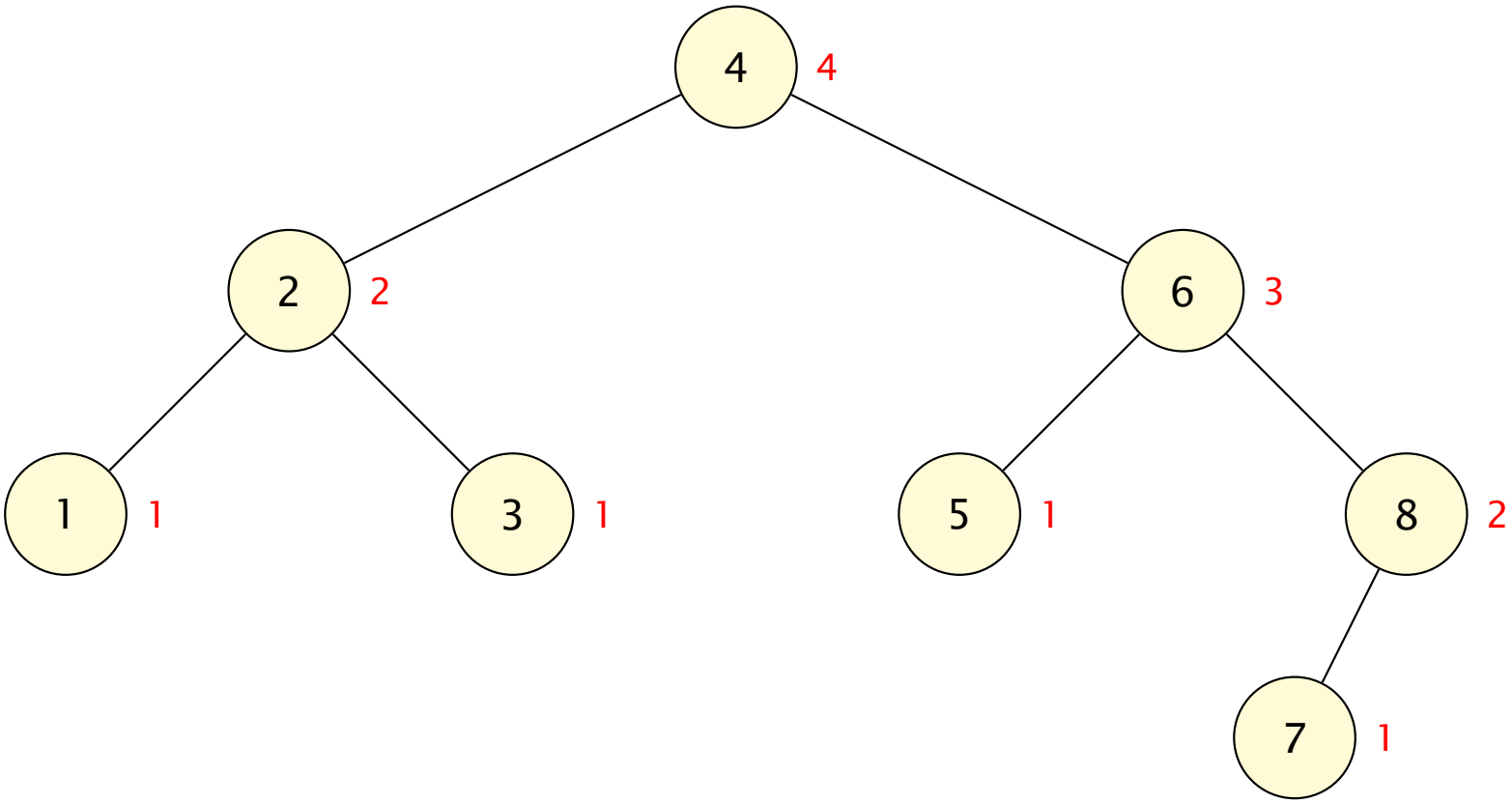
AVL Tree Delete Insertion

Answer 23 Deleting Inserted List

```
exsAVLDelListQ1a02 \
 = deleteAVLT(listQ1a[-2], exsAVLDelListQ1a01)
# Right rotation about node 8
```

exsAVLDelListQ1a02



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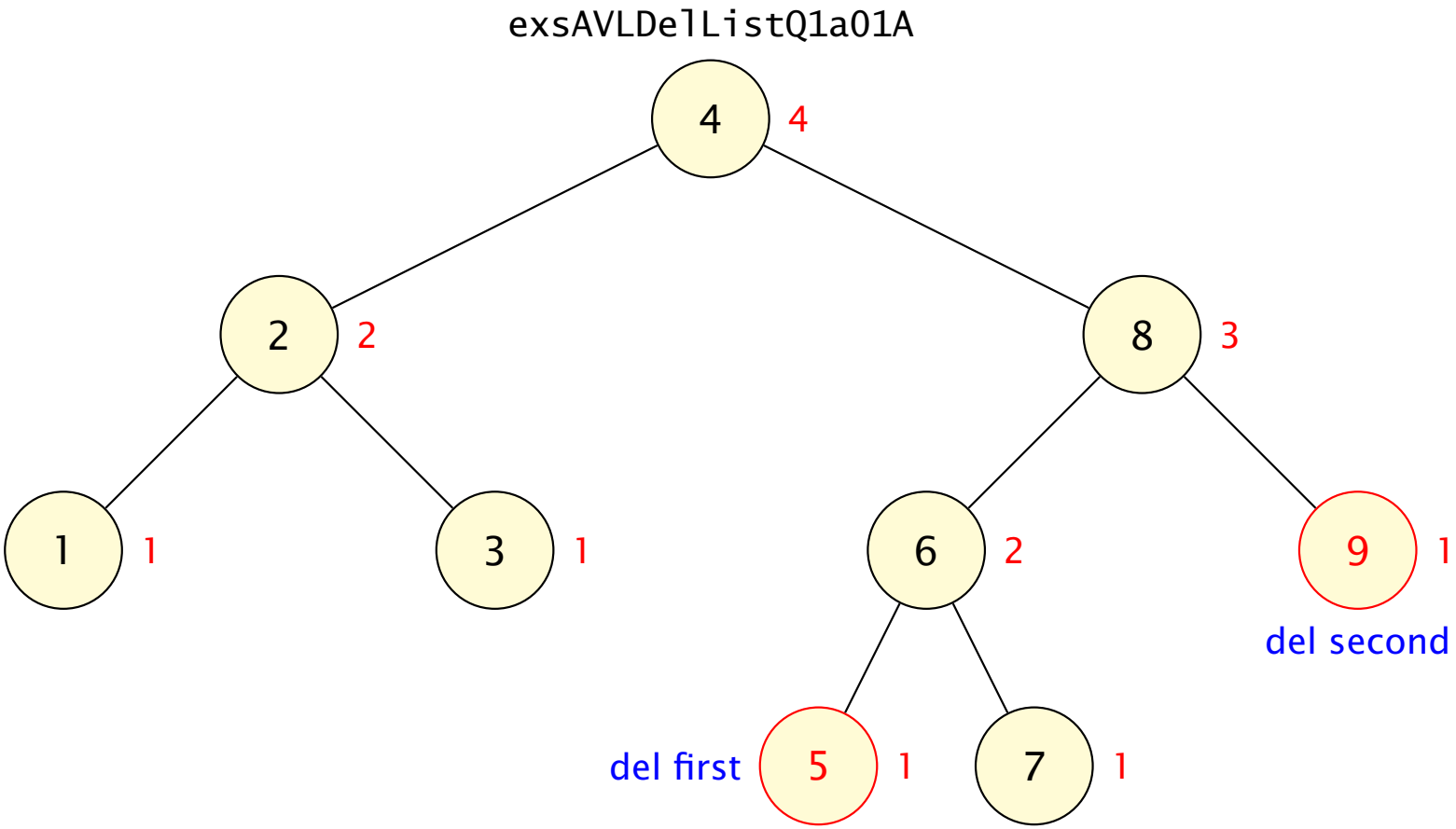
AVL Tree Delete Insertion

Answer 23 Deleting Inserted List

```
exsAVLDelListQ1a01 \
 = deleteAVLT(listQ1a[-1], exsAVLInsDelQ1a)
# delete 5 first followed by 9
```



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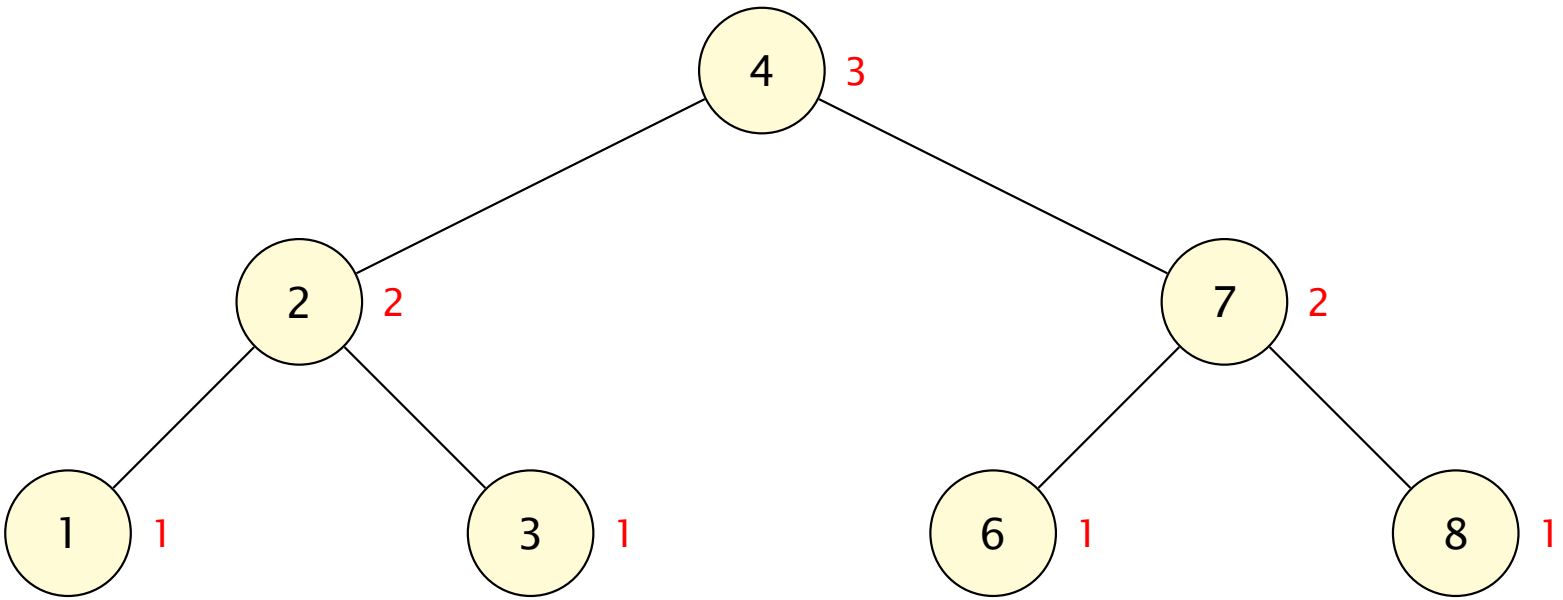
AVL Tree Delete Insertion

Answer 23 Deleting Inserted List

```
exsAVLDelListQ1a02a \
 = deleteAVLT(listQ1a[-6], exsAVLDelListQ1a01)
exsAVLDelListQ1a02b \
 = deleteAVLT(listQ1a[-2], exsAVLDelListQ1a02a)
# Double rotation: left about 6, right about 8
```

exsAVLDelListQ1a02b



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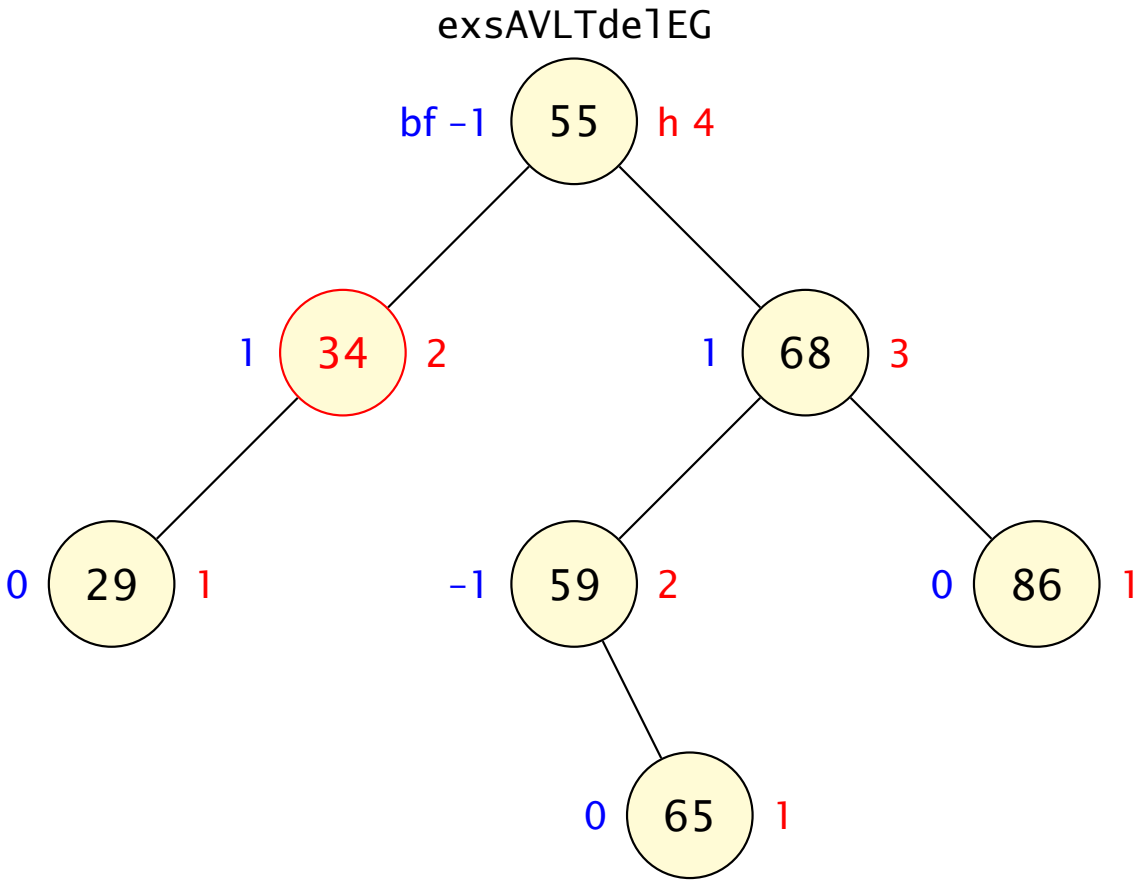
AVL Tree Delete Example

Activity 24 Delete with Rebalance

- Example from Specimen Exam (2016) Q 8
- Redraw the tree with node 34 deleted and tree rebalanced. Note here we have height of empty tree as 0 and singleton node as 1



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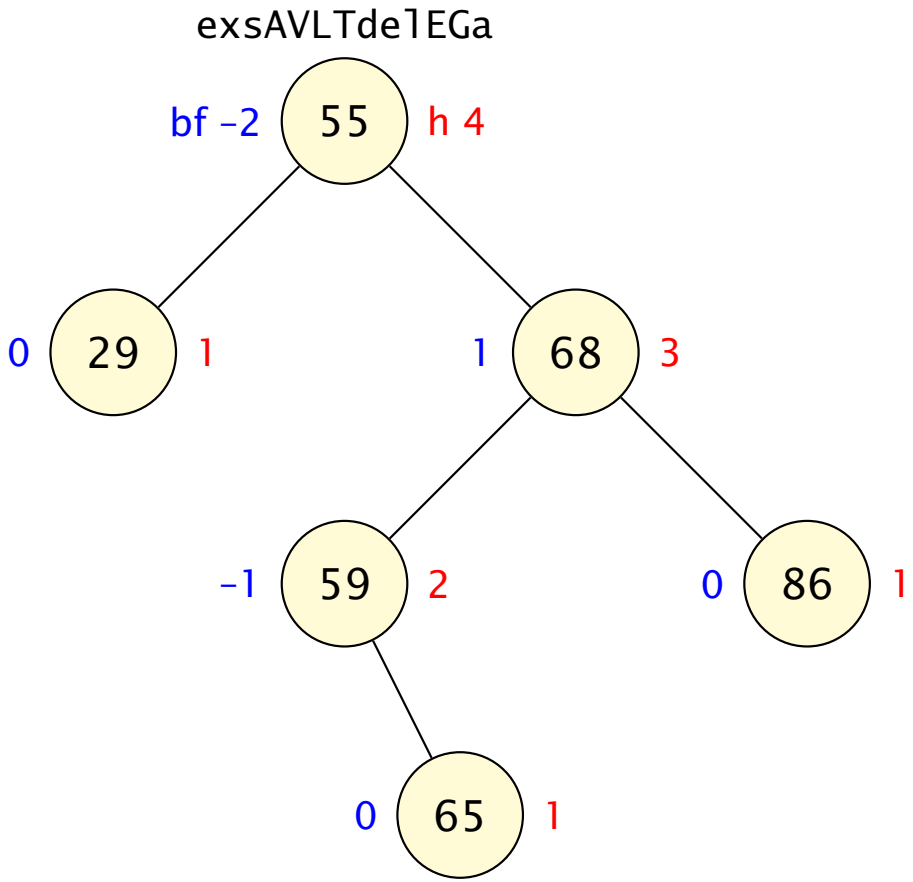
AVL Tree Delete Example

Answer 24 Delete with Rebalance (1)

- Here is the tree with node 34 deleted but not rebalanced
- The new balance factor for the root is -2 so two possible transformations — RR heavy or RL heavy



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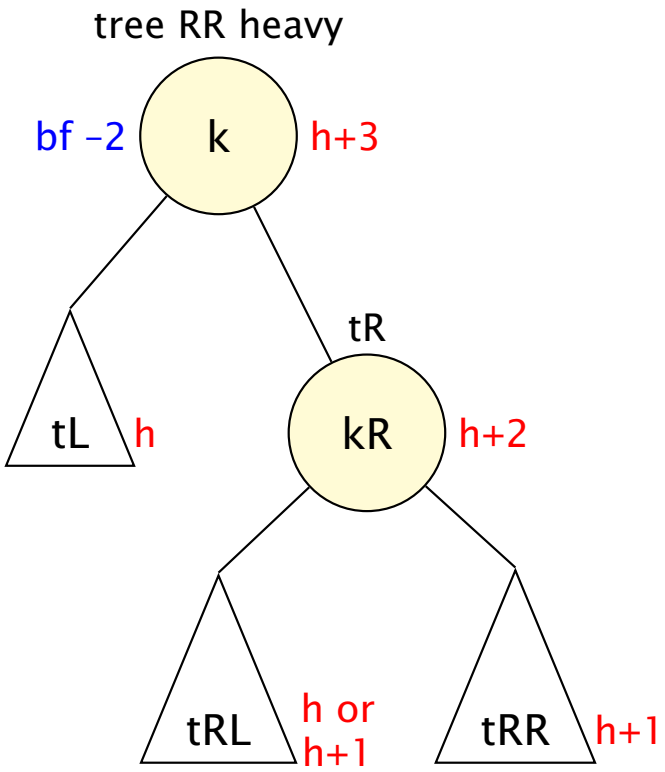
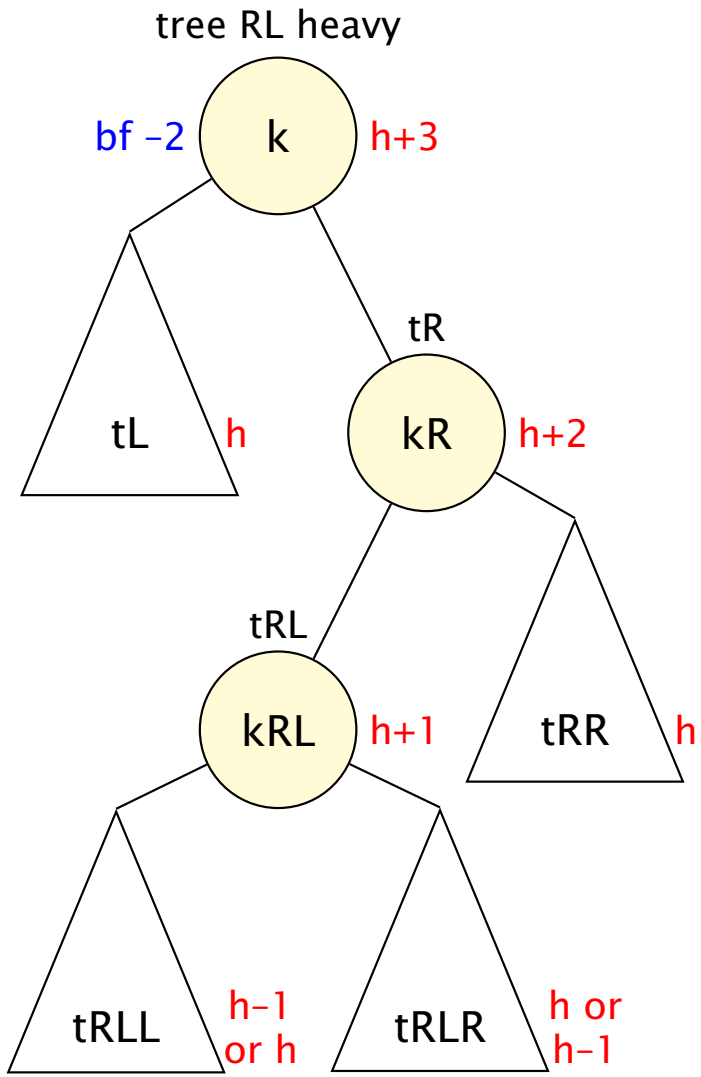
AVL Tree Delete Example

Answer 24 Delete with Rebalance (2)





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AVL Tree Delete Example

Answer 24 Delete with Rebalance (3)

- Exercise: Identify the parts of the tree given in the question with the names given for key nodes and subtrees given in the above diagrams
- Which of the two cases is the given tree an instance of ?
- Answer 24 continued on next slide

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AVL Tree Delete Example

Answer 24 Delete with Rebalance (4)

- Key k is 55
- Key kR is 68
- Key kRL is 59
- Subtree tL is rooted at 29
- Subtree tRL is rooted at 59
- Subtree RR is rooted at 86
- Subtree tRLL is an empty tree
- Subtree tRLR is rooted at 65
- The given tree is an instance of RL heavy
- This requires a double rotation to rebalance

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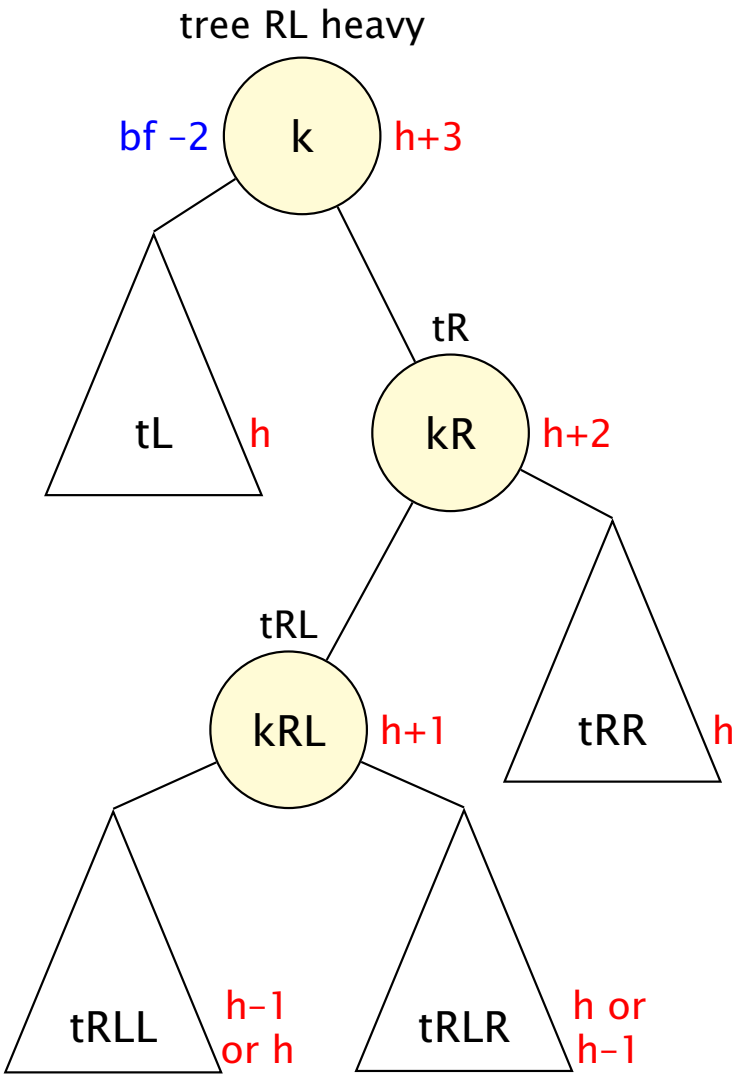
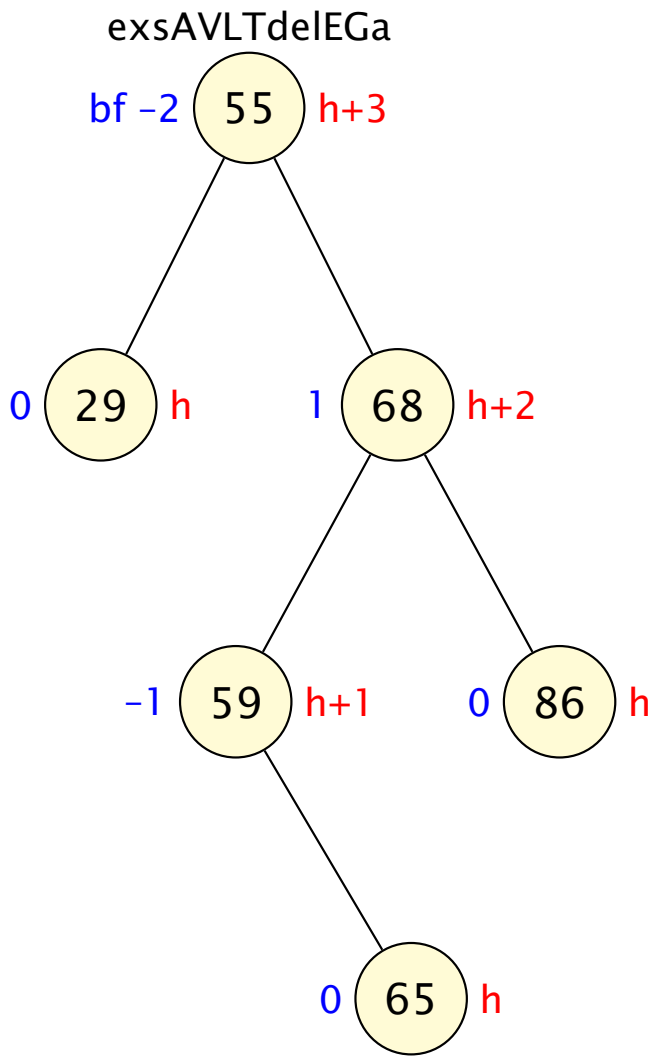
AVL Tree Delete Example

Answer 24 Delete with Rebalance (5)





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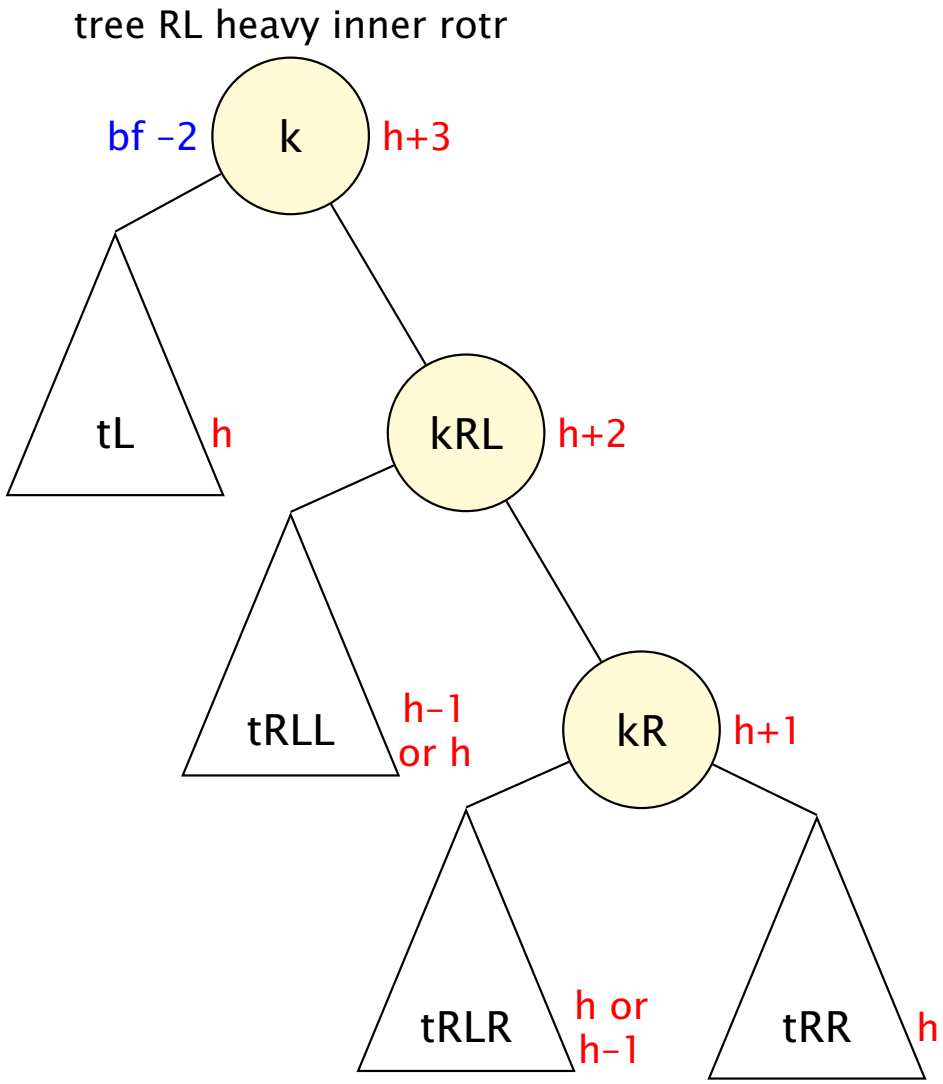
AVL Tree Delete Example

Answer 24 Delete with Rebalance (6) — Right Inner Rotation



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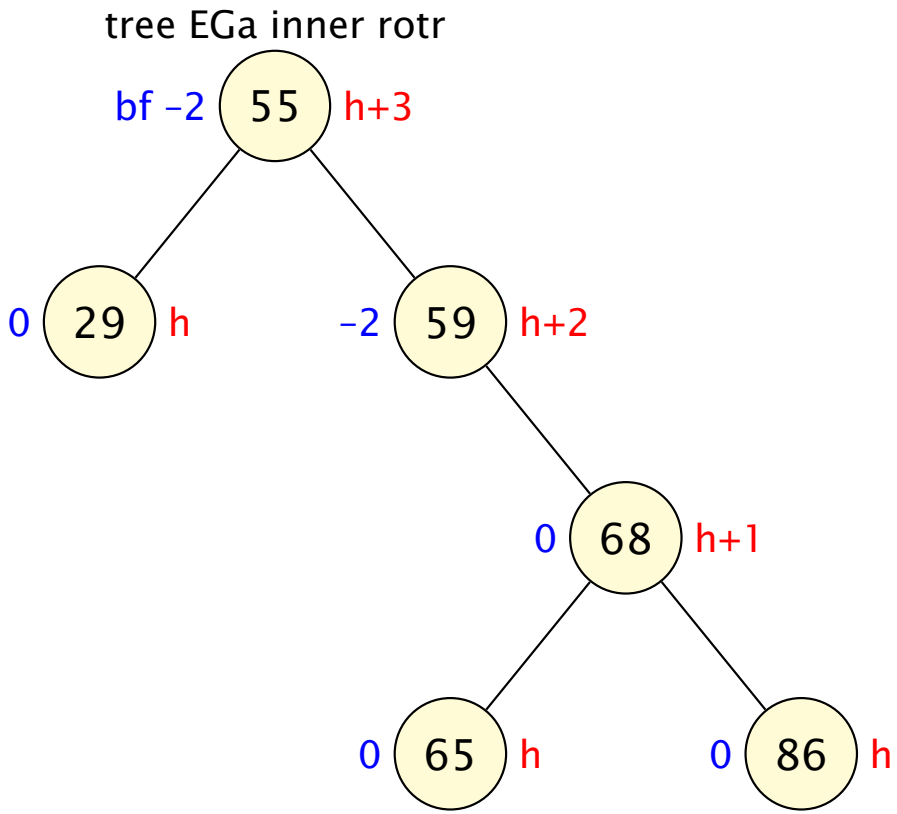
AVL Tree Delete Example

Answer 24 Delete with Rebalance (7) — Right Inner Rotation on EGa



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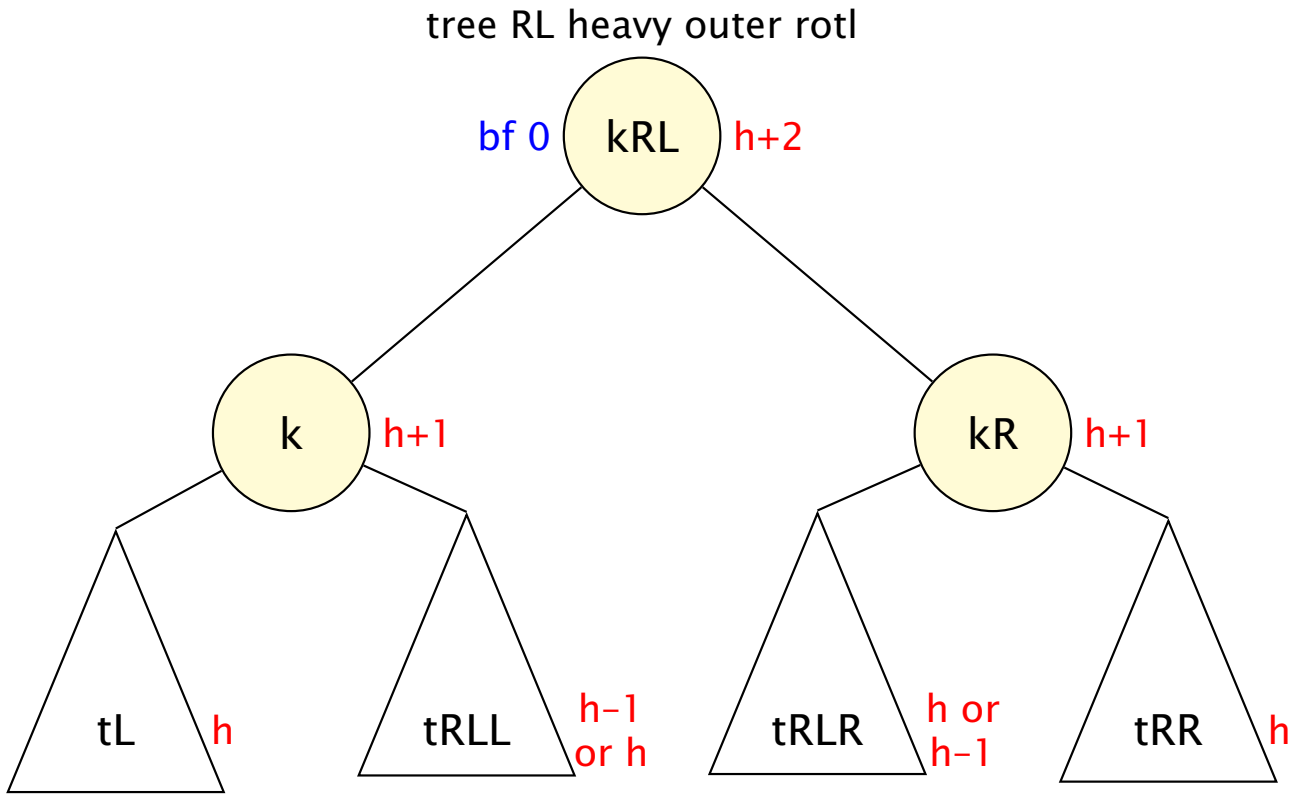
AVL Tree Delete Example

Answer 24 Delete with Rebalance (8) — Left Outer Rotation



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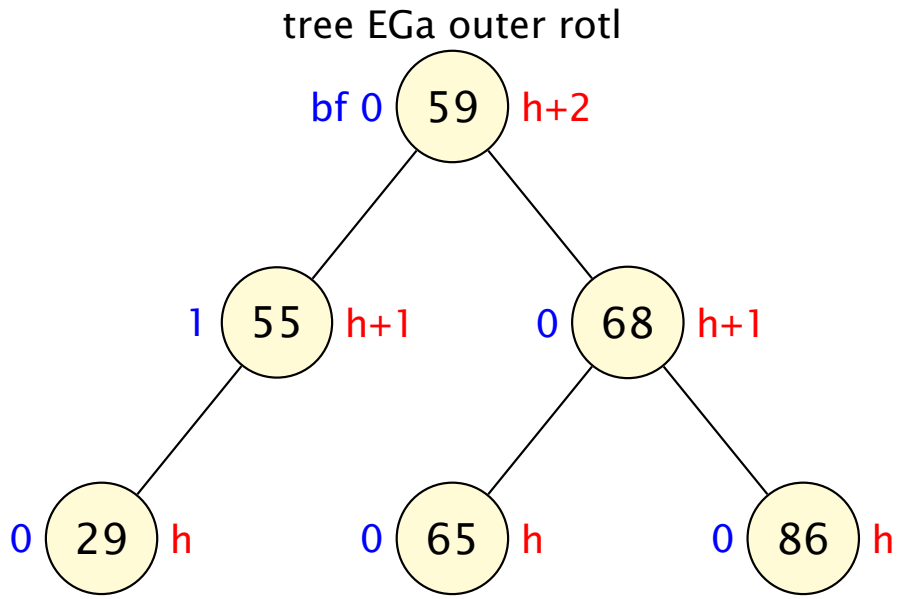
AVL Tree Delete Example

Answer 24 Delete with Rebalance (9) — Left Outer Rotation on EGa



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AVL Tree Delete Example

Answer 24 Delete with Rebalance (10) — Wrong Rotation

- Exercise: what would have happened if we had chosen only to do a left rotation around the root ?

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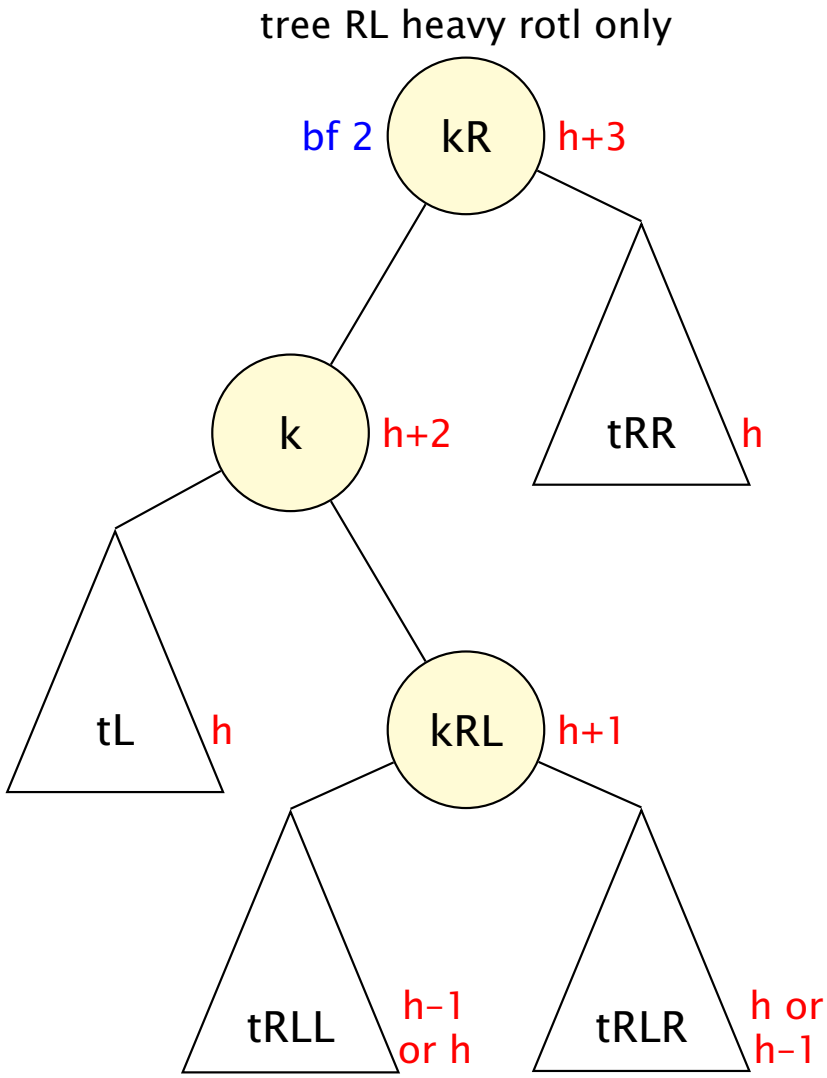
AVL Tree Delete Example

Answer 24 Delete with Rebalance (11) — Left Rotation Only



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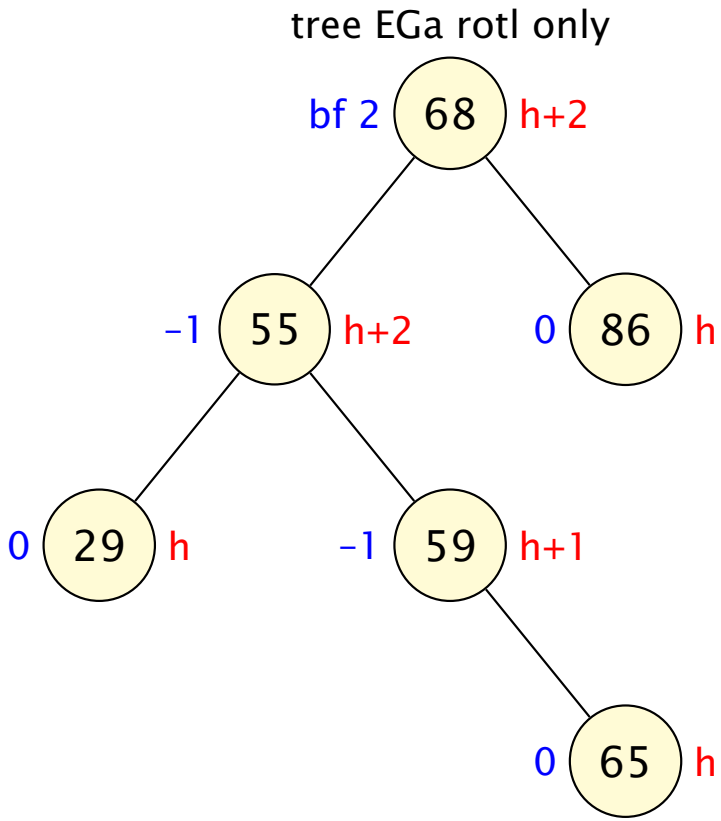
AVL Tree Delete Example

Answer 24 Delete with Rebalance (12) — Left Outer Rotation on EGa



- This tree is LR heavy and could be rebalanced via a further double rotation but obviously this would be extra work compared to getting the correct double rotation in the first place
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AVL Tree Delete Example

Answer 24 Delete with Rebalance (13)

- **Key point** when performing a rebalance, check which case applies
- **LL heavy** right rotation
- **LR heavy** inner left rotation, right outer rotation
- **RL heavy** inner right rotation, left outer rotation
- **RR heavy** left rotation
- See the notes for the details

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AVL Trees

Performance (1)

- While a height balanced tree may not always have the minimum possible height, it has the advantage that it will always be reasonably small
- For a tree with n items we shall show that the maximum number of steps to insert, delete or retrieve an item is $O(\log n)$
- Finding the maximum height of a tree with n items is equivalent to finding the minimum number of items, T_h in a tree of height h
- For $h = 0$ we have an empty tree so $T_0 = 0$
- For T_1 we have a singleton item so $T_1 = 1$
- In general for $h \geqslant 2$ we have $T_h = 1 + T_{h-1} + T_{h-2}$

  since the tree must be balanced and each subtree must have a minimum number of items

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AVL Trees

Performance (2)

- The sequence T_h looks very similar to the Fibonacci sequence
- $F_0 = 0$, $F_1 = 1$
- $F_k = F_{k-1} + F_{k-2}$, $k \geqslant 2$
- The Fibonacci sequence appeared in a work by Leonardo Fibonacci Pisano, who also popularized the Hindu-Arabic numeral system via his 1202 book *Liber Abaci (Book of Calculations)*.
- The sequence also appeared in Indian mathematics much earlier.
- The Fibonacci numbers have lots of interesting properties and turn up in many places in nature
- In our case we have $T_h = F_{h+2} - 1$

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AVL Trees

Performance (2a)

- Deriving $T_h = F_{h+2} - 1$
- Let $R_h = T_h - T_{h-1} = 1 + T_{h-2}$
- Then $R_{h+2} = 1 + T_h = 1 + 1 + T_{h-1} + T_{h-2} = R_{d+1} + R_d$
- $R_2 = 1 + T_0 = 1 + 0 = 1 = F_2$ and
  $R_3 = 1 + T_1 = 1 + 1 = 2 = F_3$
- Hence $R_h = F_h, \ \forall h \geqslant 2$
- Hence $T_h = F_{h+2} - 1, \ \forall h \geqslant 0$

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AVL Trees

Performance (2b)

- Miller & Ranum approach
- **Level** number of edges from root to node
- **Height** maximum level of any node in the tree — this is one less than my definition
- N_h is the minimum number of nodes in an AVL tree of height h
- $N_0 = 1$ since tree of one node has no edges

  $N_1 = 2$
- $N_h = 1 + N_{h-1} + N_{h-2},\ h \geqslant 2$
- Let $S_h = N_h - N_{h-1} = 1 + N_{h-2}$
- Then $S_{h+2} = 1 + N_h = 1 + 1 + N_{h-1} + N_{h-2} = S_{d+1} + S_d$
- $S_2 = 1 + N_0 = 1 + 1 = 2 = F_3$ and

  $S_3 = 1 + N_1 = 1 + 2 = 3 = F_4$
- Hence $S_h = F_{h+1},\ \forall h \geqslant 2$
- Hence $N_h = F_{h+3} - 1,\ \forall h \geqslant 0$

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AVL Trees

Performance (2c)

- $P(h) : T_h = F_{h+2} - 1$ proof by induction
- **Basis** $P(0), P(1)$
- $T_0 = 1$ and $F_2 - 1 = 1 - 1 = 0$
- $T_1 = 1$ and $F_3 - 1 = 2 - 1 = 1$
- **Inductive step** $\forall k\, P(k) \Rightarrow P(k+1)$
- $$\begin{aligned} T_k &= 1 + T_{k-1} + T_{k-2} \\ &= 1 + (F_{k+1} - 1) + (F_k - 1) \\ &= F_{k+2} - 1 \end{aligned}$$
- Hence $T_h = F_{h+2} - 1, \ \forall h \geqslant 0$

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AVL Trees

Performance (3)

- There is a closed-form solution for the Fibonacci sequence known as the Euler-Binet Formula (see also A formula for Fib(n))
- $F_k = \frac{\phi^k - (1-\phi)^k}{\sqrt{5}}$
- ϕ is the Golden mean
- $\phi = \frac{1}{\phi - 1} = \frac{1+\sqrt{5}}{2} \approx 1.61803\ldots$
- Hence $T_h = \frac{\phi^{h+2} - (1-\phi)^{h+2}}{\sqrt{5}} - 1$
- Since $(1-\phi) < 1$ then for large h we have
- $T_h = n \approx \frac{\phi^{h+2}}{\sqrt{5}} - 1 \rightarrow \log(\sqrt{5}(n+1)) \approx (h+2)\log\phi$
- Hence in the worst case, the height of a AVL tree is $O(\log n)$

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Fibonacci Euler-Binet Formula

Proof (1a)

- Proof of the Euler-Binet Formula is not required for M269 but here is a brief summary
- **Proof by Induction**
- Let $P(n) = F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$
- **Basis for Induction**
- $P(0)$ is true since

$$\frac{\phi^0 - (1-\phi)^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 == F_0$$

- $P(1)$ is true since

$$\frac{\phi^1 - (1-\phi)^1}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right) - \left(1 - \left(\frac{1+\sqrt{5}}{2}\right)\right)}{\sqrt{5}}$$

- $= 1 == F_1$

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Fibonacci Euler-Binet Formula

Proof (1b)

- **Induction Hypothesis Step**
- Show $P(j) : 0 \leqslant j \leqslant k+1 \Rightarrow P(k+2)$
- $\phi^{k+2} - (1-\phi)^{k+2} = \phi^2\phi^k - (1-\phi)^2(1-\phi)^k$ and
- $\phi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1}{4}(1+2\sqrt{5}+5) = 1+\phi$
- $(1-\phi)^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2 = 1+(1-\phi)$ hence
- $\phi^{k+2} - (1-\phi)^{k+2} = (1+\phi)\phi^k - (1+(1-\phi))(1-\phi)^k$
- $= \left(\phi^k - (1-\phi)^k\right) + \left(\phi^{k+1} - (1-\phi)^{k+1}\right)$
- $= \sqrt{5}(F_k + F_{k+1})$ by inductive hypothesis
- $= \sqrt{5}F_{k+2}$ by Fibonacci definition
- Hence $\forall n \in \mathbb{N} : F_n = \dfrac{\phi^n - (1-\phi)^n}{\sqrt{5}}$

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Fibonacci Euler-Binet Formula

Proof (2a)

- The above proof confirms the formula but here is a derivation
- Define $T(x, y) = (y, x + y)$
- Then $T^n(0, 1) = (F_n, F_{n+1})$ (proof by induction)
- Now find λ_1, λ_2 and $(x_1, y_1), (x_2, y_2)$

  so $T(x_1, y_1) = \lambda_1(x_1, y_1)$ and $T(x_2, y_2) = \lambda_2(x_2, y_2)$

  and $(0, 1) = p_1(x_1, y_1) + p_2(x_2, y_2)$
- $T(x, y) = (y, x + y) = \lambda(x, y)$

  $\rightarrow x + y = \lambda x$ and $x = \lambda y \rightarrow \lambda^2 - \lambda - 1 = 0$

  $\rightarrow \lambda_1 = \phi$ and $\lambda_2 = 1 - \phi$

  and $T(1, \phi) = (\phi, 1 + \phi) = \phi(1, \phi)$

  $T(1, 1 - \phi) = (1 - \phi, 1 + (1 - \phi)) = (1 - \phi)(1, 1 - \phi)$

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Fibonacci Euler-Binet Formula

Proof (2b)

- $(0,1) = \frac{1}{\sqrt{5}}(1,\phi) - \frac{1}{\sqrt{5}}(1,1-\phi)$ confirm by inspection
- $(F_n, F_{n+1}) = T^n(0,1)$
  $= \frac{1}{\sqrt{5}} T^n(1,\phi) - \frac{1}{\sqrt{5}} T^n(1,1-\phi)$
  $= \frac{1}{\sqrt{5}} \phi^n(1,\phi) - \frac{1}{\sqrt{5}}(1-\phi)^n(1,1-\phi)$
  $= \frac{1}{\sqrt{5}}(\phi^n, \phi^{n+1}) - \frac{1}{\sqrt{5}}((1-\phi)^n, (1-\phi)^{n+1})$
- Hence $F_n = \dfrac{\phi^n - (1-\phi)^n}{\sqrt{5}}$

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AVL Tree Application

Sets

- Ordered sets and ordered maps are important data types in programming
- Some programming languages have them as builtin types (Python) or supply them as standard libraries (C++, C#, Java, Scala, Haskell, ML)
- This section describes an example implementation based on Blelloch et al (2016) *Just Join for Parallel Ordered Sets* and Adams (1993) *Functional Pearls Efficient sets — a balancing act*
- Note that this example also shows the use of recursive thinking in practice

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AVL Tree Application

Documentation

- In Python the documentation for Sets is at Set Types and for Dictionaries (Maps) at Mapping Types — dict
- The Python implementation can be found at the Python Developer's Guide and the source code for Sets is at setobject.c — the implementation is in C using hash tables — see How is set() implemented?
- In Haskell the documentation and implementation of Sets is at Containers: Data.Set and for Maps at Containers: Data.Map.Strict — both of these are from the package containers: Assorted concrete container types
- The Haskell implementation uses size balanced trees — this is similar to AVL balanced trees
- For an introduction see containers - Introduction and Tutorial
- For an overview of Sets see Containers: Sets

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AVL Tree Application

Sets

- M269 Unit 5 has representation of graphs for various algorithms — here are some references for that topic for future notes
- For Haskell graph libraries see
  - fgl: Martin Erwig's Functional Graph Library
  - graphs: A simple monadic graph library by Edward Kmett
  - Data.Graph based on King and Launchbury (1995) *Structuring depth-first search algorithms in Haskell*

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AVL Trees

Set Representation — Split, Join (1)

- The representation of sets uses the `ABTree` data type but with generalised versions of the `split` and `join` functions
- While implementing sets in AVL trees is not directly part of M269, it gives good examples of recursive thinking in an important application
- Note that the `data` item is used as the key for a node in the tree — in practice there would be separate key and data

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AVL Trees

Set Representation — Split, Join (2)

- `splitAVLS` takes a key `k` and an AVL tree `t` and returns two trees `tL` and `tR` and a boolean `b` — `tL` and `tR` have elements less than and greater than `k` respectively and `b` indicates if `k` was in `t`
- `splitLastAVLS`, `splitFirstAVLS` take an AVL tree `t` and return the largest, smallest elemeent `k` respectively and the rest of the tree — `splitFirstAVLS` is similar to `splitAVLT` at line 926 on slide 223
- `join2AVLS`, `joinAVLS` take two AVL trees, `tL`, `tR` where all elements of `tL` are less than all elements of `tR` and returns a new AVL tree — `joinAVLS` also takes a key `k` with a value in between the elements of the two trees — `join2AVLS` is similar to `joinAVLT` at line 917 on slide 223
- `exposeABT` takes apart an augmented tree, `t` to give `(tL, k, tR)`

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AVL Trees

Set Representation — Split, Join (3)

- Here is a reminder of some of the ABTree constructors and inspectors from file M269TutorialBinaryTrees2022.py

```
761 def mkEmptyABT() -> ABTree :
762   return EmptyABT()

764 def mkNodeABT(x: T,t1: ABTree,t2: ABTree) -> ABTree :
765   h = 1 + max(getHeightABT(t1),getHeightABT(t2))
766   return NodeABT(h,x,t1,t2)

768 def isEmptyABT(t: ABTree) -> bool :
769   return t == EmptyABT()
```

- And here is the additional inspector expose in M269TutorialBinaryTrees2022AVLSets.py

```
11 def exposeABT(t: ABTree) -> (ABTree, T, ABTree) :
12   if isEmptyABT(t) :
13     raise RuntimeError("exposeABT␣applied␣to␣EmptyABT()")
14   else :
15     tL = getLeftABT(t)
16     k  = getDataABT(t)
17     tR = getRightABT(t)
18     return (tL,k,tR)
```


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AVL Trees

Set Representation — Split, Join (4)

- joinAVLS take a key, k, two AVL trees, tL, tR where all elements of tL are less than k which is less than all elements of tR and returns a new AVL tree

```
20 def joinAVLS(k: T, tL: ABTree, tR: ABTree) -> ABTree :
21   if getHeightABT(tL) > getHeightABT(tR) + 1 :
22     return joinRightAVLS(k,tL,tR)
23   elif getHeightABT(tR) > getHeightABT(tL) + 1 :
24     return joinLeftAVLS(k,tL,tR)
25   else :
26     return mkNodeABT(k,tL,tR)
```


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AVL Trees

Set Representation — Split, Join (5)

- joinRightAVLS description is in the following diagrams

```
28 def joinRightAVLS(k: T, tL: ABTree, tR: ABTree) -> ABTree :
29   (tLL,kL,tLR) = exposeABT(tL)
30   if getHeightABT(tLR) <= getHeightABT(tR) + 1 :
31     t1 = mkNodeABT(k,tLR,tR)
32     if getHeightABT(t1) <= getHeightABT(tLL) + 1 :
33       return mkNodeABT(kL,tLL,t1)
34     else :
35       return rotl(mkNodeABT(kL,tLL,(rotr(t1))))
36   else :
37     t2 = joinRightAVLS(k,tLR,tR)
38     t3 = mkNodeABT(kL,tLL,t2)
39     if getHeightABT(t2) <= getHeightABT(tLL) + 1 :
40       return t3
41     else :
42       return rotl(t3)
```


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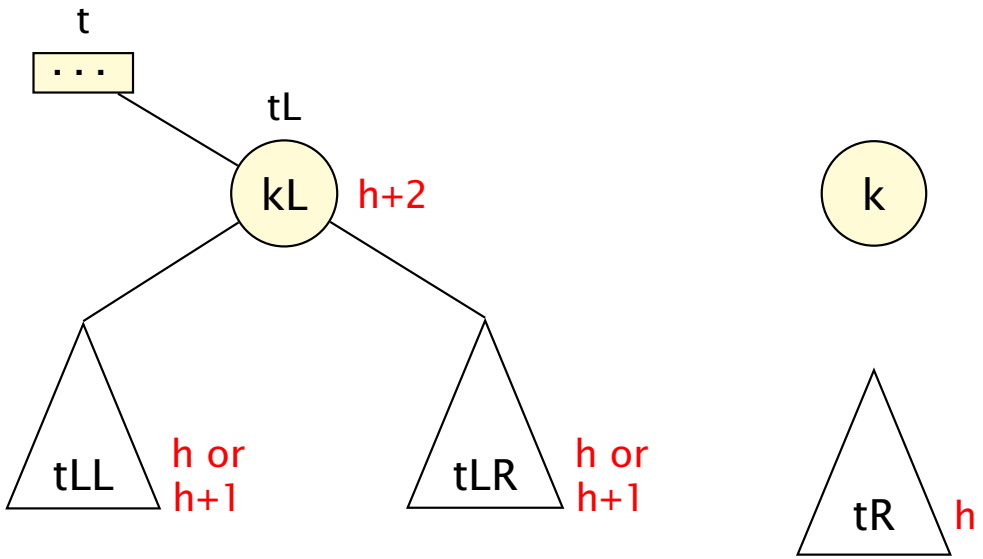
Set Representation — Split, Join (7)



- The base case (line 30 on slide 279) of `joinRightAVLS` follows the right spine of `t` to a node `kL` for which

```
getHeightABT(tL) > getHeightABT(tR) + 1
getHeightABT(tLR) <= getHeightABT(tR) + 1
```

- We then connect `tL`, `k` and `tR`

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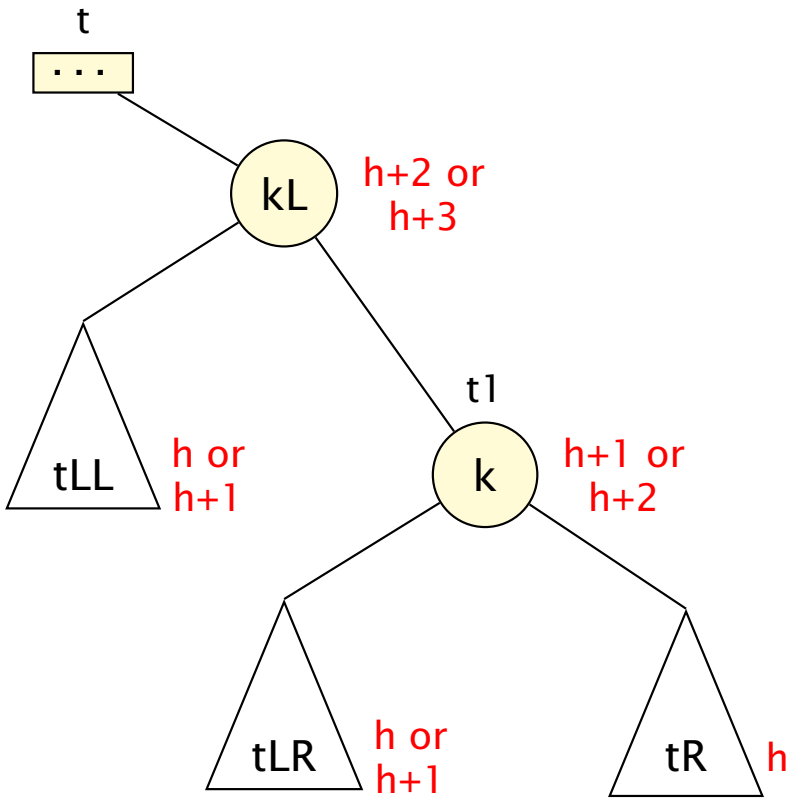
Set Representation — Split, Join (8)



- Needs double rotation if

```
getHeightABT(t1) > getHeightABT(tLL) + 1
```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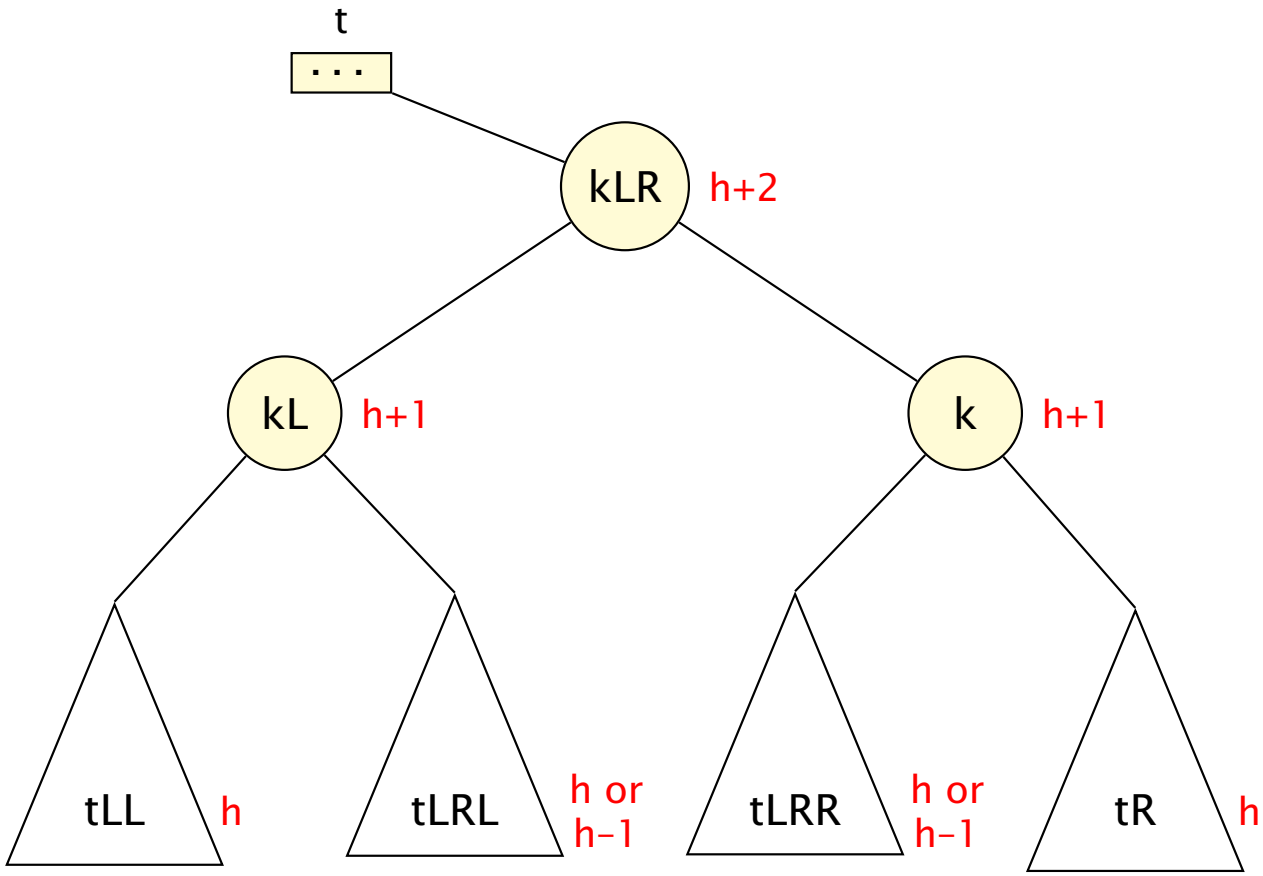
Set Representation — Split, Join (9)



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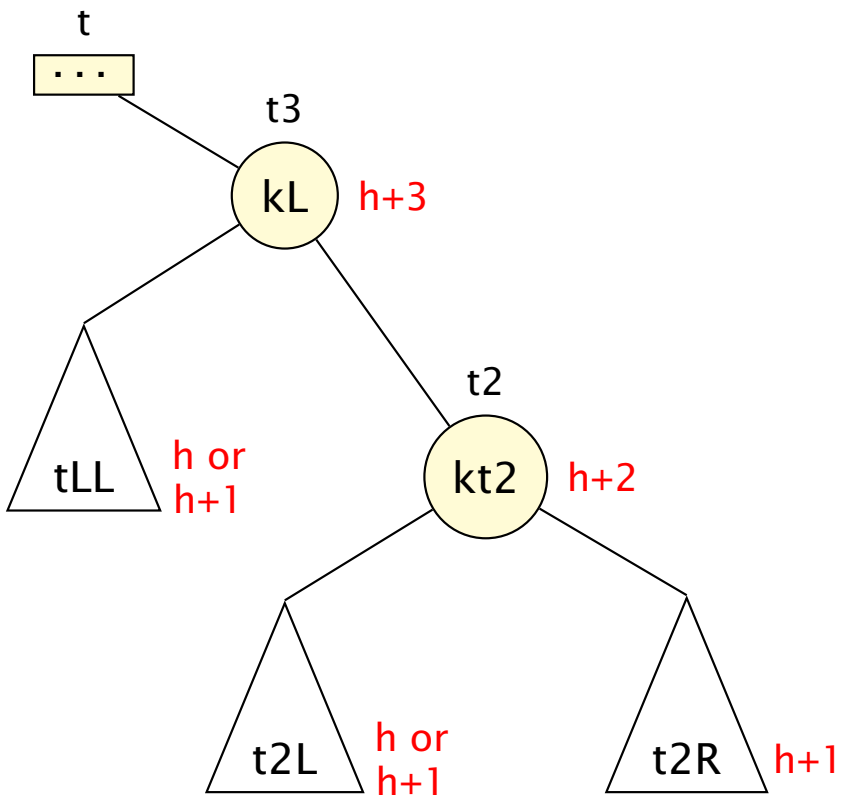
Set Representation — Split, Join (10)

- The recursive case (line 36 on slide 279) of `joinRightAVLS` follows the right spine further

```
getHeightABT(tLR) > getHeightABT(tR) + 1
```



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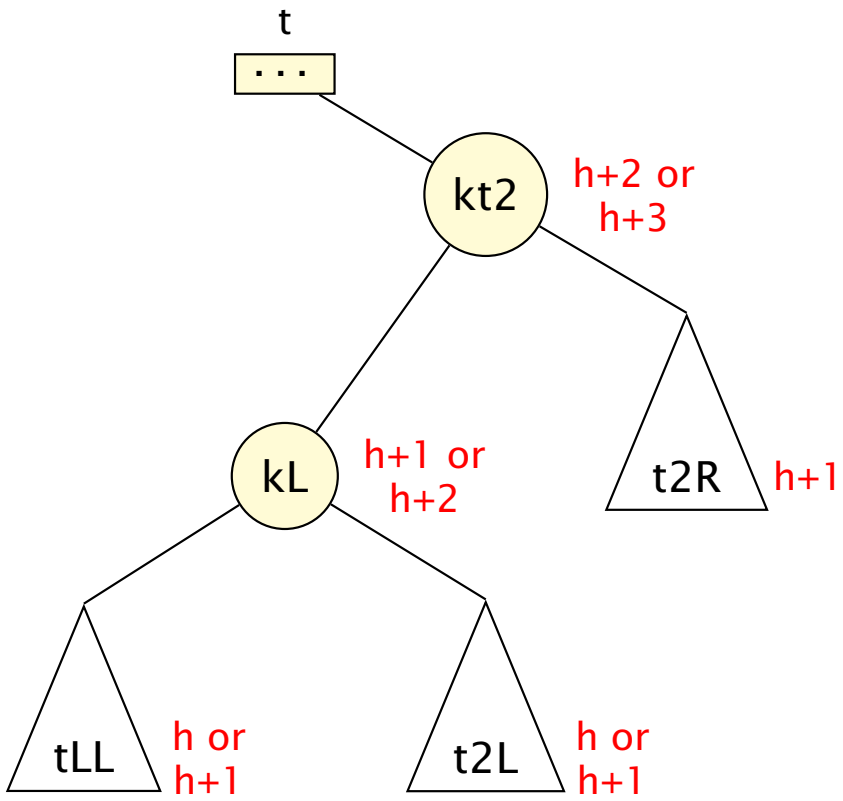
AVL Trees

Set Representation — Split, Join (11)

- A single left rotation is needed if

```
getHeightABT(t2) > getHeightABT(tLL) + 1
```



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AVL Trees

Set Representation — Split, Join (12)

```
44 def joinLeftAVLS(k: T, tL: ABTree, tR: ABTree) -> ABTree :
45   (tRL, kR, tRR) = exposeABT(tR)
46   if getHeightABT(tRL) <= getHeightABT(tL) + 1 :
47     t1 = mkNodeABT(k,tL,tRL)
48     if getHeightABT(t1) <= getHeightABT(tRR) + 1 :
49       return mkNodeABT(kR,t1,tRR)
50     else :
51       return rotr(mkNodeABT(kR,(rotl(t1)),tRR))
52   else :
53     t2 = joinLeftAVLS(k,tL,tRL)
54     t3 = mkNodeABT(kR,t2,tRR)
55     if getHeightABT(t2) <= getHeightABT(tRR) + 1 :
56       return t3
57     else :
58       return rotr(t3)
```


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Activity 25 joinLeftAVLS Diagrams

- `joinLeftAVLS` is the mirror image of `joinRightAVLS`
- Produce the equivalent diagrams describing the function

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AVL Trees

Answer 25 joinLeftAVLS Diagrams

- TODO: Answer 25 joinLeftAVLS Diagrams

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Activity 26 joinLeftAVLS Bug

- A previous version of `joinLeftAVLS` had a bug (beware copy/paste) — see below
- What would happen if the elements of [10,9,8,7,6] were given as input ?

```
def joinLeftAVLS(k: T, tL: ABTree, tR: ABTree) -> ABTree :
  (tRL, kR, tRR) = exposeABT(tR)
  if getHeightABT(tRL) <= getHeightABT(tL) + 1 :
    t1 = mkNodeABT(k,tL,tRL)
    if getHeightABT(t1) <= getHeightABT(tRR) + 1 :
      return mkNodeABT(kR,t1,tRR)
    else :
      return rotr(mkNodeABT(kR,(rotl(t1)),tRR))
  else :
    t2 = joinRightAVLS(k,tL,tRL)
    t3 = mkNodeABT(kR,t2,tRR)
    if getHeightABT(t2) <= getHeightABT(tRR) + 1 :
      return t3
    else :
      return rotr(t3)
```

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Answer 26 joinLeftAVLS Bug

- TODO: Answer 26 joinLeftAVLS Bug

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AVL Trees

Set Representation — Split, Join (13)

```
60 def splitLastAVLS(t: ABTree) -> (ABTree,T) :
61   (tL,k,tR) = exposeABT(t)
62   if isEmptyABT(tR) :
63     return (tL,k)
64   else :
65     (tR1,k1) = splitLastAVLS(tR)
66     return (joinAVLS(k,tL,tR),k1)

68 def splitFirstAVLS(t: ABTree) -> (ABTree,T) :
69   (tL, k, tR) = exposeABT(t)
70   if isEmptyABT(tL) :
71     return (tR,k)
72   else :
73     (tL1,k1) = splitFirstAVLS(tL)
74     return (joinAVLS(k,tL1,tR),k1)
```


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AVL Trees

Set Representation — Split, Join (14)

```
76 def splitAVLS(k: T,t: ABTree) -> (ABTree,bool,ABTree) :
77   if isEmptyABT(t) :
78     return (mkEmptyABT(),False,mkEmptyABT())
79   else :
80     (tL, k1, tR) = exposeABT(t)
81     if k == k1 :
82       return (tL,True,tR)
83     elif k < k1 :
84       (tLL, b, tLR) = splitAVLS(k,tL)
85       return (tLL, b, (joinAVLS(k1,tLR,tR)))
86     else :
87       (tRL, b, tRR) = splitAVLS(k,tR)
88       return ((joinAVLS(k1,tL,tRL)), b, tRR)
```


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AVL Trees

Set Representation — Split, Join (15)

```
90 def join2AVLS(tL: ABTree,tR: ABTree) -> ABTree :
91   if isEmptyABT(tL) :
92     return tR
93   else :
94     (tL1, k) = splitLastAVLS(tL)
95     return joinAVLS(k,tL1,tR)
```


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AVL Trees

Set Operations (1)

- **Set Operations**
- `insertAVLS(t,k)` inserts a key, `k`, into a tree, `t`
- `deleteAVLS(t,k)` deletes key, `k`, from a tree, `t`, if it is in the tree
- `unionAVLS(t1,t2)` takes two AVL trees whose values may overlap, and returns the union as a tree
- `intersectAVLS(t1,t2)` takes two AVL trees and returns the intersection as a tree
- `disjoint(t1,t2)` takes two AVL trees and returns `True` if and only if they have no members in common
- `differenceAVLS t1 t2` takes two AVL trees and returns the elements that are in `t1` but not `t2`
- `subsetAVLS(t1,t2)` takes two AVL trees and returns `True` if and only if every member of `t1` is a member of `t2`

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AVL Trees

Set Operations (2)

```
97 def insertAVLS(t: ABTree,k: T) -> ABTree :
98   (tL, found, tR) = splitAVLS(k,t)
99   return joinAVLS(k,tL,tR)

101 def deleteAVLS(t: ABTree,k: T) -> ABTree :
102   (tL, found, tR) = splitAVLS(k,t)
103   return join2AVLS(tL,tR)

105 def insertListAVLS(t: ABTree,xs: [T]) -> ABTree :
106   if xs == [] :
107     return t
108   else :
109     return insertListAVLS(insertAVLS(t,xs[0]),xs[1:])

111 def setFromListAVLS(xs: [T])-> ABTree :
112   return insertListAVLS(mkEmptyABT(),xs)
```


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AVL Trees

Set Operations (3)

```
114 def unionAVLS(t1: ABTree,t2: ABTree) -> ABTree :
115   if   isEmptyABT(t1) :
116     return t2
117   elif isEmptyABT(t2) :
118     return t1
119   else :
120     (t2L, k2, t2R) = exposeABT(t2)
121     (t1L, found, t1R) = splitAVLS(k2,t1)
122     tL = unionAVLS(t1L,t2L)
123     tR = unionAVLS(t1R,t2R)
124     return joinAVLS(k2,tL,tR)
```

- `unionAVLS(t1,t2)` returns the set of all members of `t1` or `t2` (or both)

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AVL Trees

Set Operations (4)

```
126 def intersectAVLS(t1: ABTree,t2: ABTree) -> ABTree :
127   if   isEmptyABT(t1) :
128     return mkEmptyABT()
129   elif isEmptyABT(t2) :
130     return mkEmptyABT()
131   else :
132     (t2L, k2, t2R) = exposeABT(t2)
133     (t1L, found, t1R) = splitAVLS(k2,t1)
134     tL = intersectAVLS(t1L,t2L)
135     tR = intersectAVLS(t1R,t2R)
136     if found :
137       return joinAVLS(k2,tL,tR)
138     else :
139       return join2AVLS(tL,tR)
```

- `intersectAVLS(t1,t2)` returns the set of all members of both `t1` and `t2`
- Notice it needs the `if` statement to check that a member of `t2` is a member of `t1`

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AVL Trees

Set Operations (5)

```
141 def disjointAVLS(t1: ABTree,t2: ABTree) -> ABTree :
142   if   isEmptyABT(t1) :
143     return True
144   elif isEmptyABT(t2) :
145     return True
146   else :
147     (t2L, k2, t2R) = exposeABT(t2)
148     (t1L, found, t1R) = splitAVLS(k2,t1)
149     return (not found
150             and disjointAVLS(t1L,t2L)
151             and disjointAVLS(t1R,t2R))
```

- `disjoint(t1,t2)` returns `True` if there are no elements in common
- If an element in common is found then `False` is returned
- Note that the behaviour of `splitAVLS()` ensures the search space is reduced at each recursive call

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AVL Trees

Set Operations (6)

```
153 def differenceAVLS(t1: ABTree,t2: ABTree) -> ABTree :
154   if   isEmptyABT(t1) :
155     return mkEmptyABT()
156   elif isEmptyABT(t2) :
157     return t1
158   else :
159     (t2L, k2, t2R) = exposeABT(t2)
160     (t1L, found, t1R) = splitAVLS(k2,t1)
161     tL = differenceAVLS(t1L,t2L)
162     tR = differenceAVLS(t1R,t2R)
163     return join2AVLS(tL,tR)
```

- `differenceAVLS(t1,t2)` returns the set of members of `t1` that are not in `t2`
- On first reading it may be surprising there is no `if` statement
- Remember the behaviour of `splitAVLS()`

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Further Set Operations

Activity 27 Set to Ascending List

- Write a function `setToAscList` which takes a set and returns the contents as an ascending list

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Set to Ascending List

Answer 27 Set to Ascending List (1)

- Probably the simplest solution at this stage is to use `inOrderABT()`

```
181 def setToAscList(t) :
182   return inOrderABT(t)
```

```
Python3>>> list3 = [2,1,4,3,6,5,8,7,10,9]
Python3>>> t10 = setFromListAVLS(list3)
Python3>>> type(t10)
<class 'M269TutorialBinaryTrees2022.NodeABT'>
Python3>>> list4 = setToAscList(t10)
Python3>>> list4
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
Python3>>>
```

- Answer 27 continued on next slide

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Set to Ascending List

Answer 27 Set to Ascending List (2)

- Of course, someone from the pure functional programming world would define a higher order function to capture the recursion pattern with `setFoldr()` (this is not part of M269)

```
184 def setFoldr(f,z,t) :
185   def go(y,t) :
186     if isEmptyABT(t) :
187       return y
188     else :
189       (tL,x,tR) = exposeABT(t)
190       return (go(f(x,(go(y,tR))),tL))
191   return go(z,t)

193 def setToAscListA(t) :
194   def cons(x,xs) :
195     return ([x] + xs)
196   return setFoldr(cons,[],t)
```

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Activity 28 Set Equality

- Write a function `setEquality` which takes two sets, `t1` and `t2` and returns `True` if they are equal and `False` otherwise

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Further Set Operations

Answer 28 Set Equality

- Answer 28 Set Equality

```
198 def setEquality(t1,t2) :
199   list1 = setToAscList(t1)
200   list2 = setToAscList(t2)
201   return (len(list1) == len(list2)
202           and list1 == list2)
```

```
Python3>>> list1
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
Python3>>> t1 = setFromListAVLS(list1)
Python3>>> list10
[2, 1, 4, 3, 6, 5, 8, 7, 10, 9]
Python3>>> t10 = setFromListAVLS(list10)
Python3>>> t1 == t10
False
Python3>>> setEquality(t1,t10)
True
Python3>>>
```

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Activity 29 Subset

- Write a function `subsetAVLS` that takes two sets `t1`, `t2` and returns `True` if `t1` is a subset of `t2` and `False` otherwise
- `t1` is a subset of `t2` if every element of `t1` is a member of `t2`

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Further Set Operations

Answer 29 Subset

```
165 def subsetAVLS(t1: ABTree,t2: ABTree) -> bool :
166   if   isEmptyABT(t1) :
167     return True
168   elif isEmptyABT(t2) :
169     return False
170   else :
171     (t1L, k1, t1R) = exposeABT(t1)
172     (t2L, found, t2R) = splitAVLS(k1,t2)
173     return (found
174             and subsetAVLS(t1L,t2L)
175             and subsetAVLS(t1R,t2R))
```

- How does this work ?
- The recursive case at line 170 checks that the key at the root of `t1` is in `t2` and recursively checks the sub-trees
- `splitAVLS()` ensures that the subtrees are the correct ones to be checked

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Sets, Maps

Implementation Points

- The only tree specific functions are `joinAVLS`, `joinRightAVLS` and `joinLeftAVLS` — AVL trees could be changed to size balanced or Red-Black trees with little to be changed
- The various sets operations use `splitAVLS` and `joinAVLS` or `join2AVLS` to avoid more complex algorithms — some implementations may inline the functions for efficiency
- The diagrams for `joinRightAVLS` are essential for the understanding of the base and recursive cases
- The `unionAVLS`, `intersectAVLS` and `differenceAVLS` functions are very similar in their usage of `split` and `join`

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Sets, Maps

Implementation Points

- From O'Sullivan *Real World Haskell (2008)* see Data Structures
- *Maps give us the same capabilities as hash tables do in other languages. Internally, a map is implemented as a balanced binary tree. Compared to a hash table, this is a much more efficient representation in a language with immutable data. This is the most visible example of how deeply pure functional programming affects how we write code: we choose data structures and algorithms that we can express cleanly and that perform efficiently, but our choices for specific tasks are often different [from] their counterparts in imperative languages.*
- See Curious about the HashTable performance issues

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Sets, Maps

Implementation Points

	Python		Haskell
Operation	**Average**	**Worst**	**Worst**
Member	$O(1)$	$O(n)$	$O(\log n)$
Union	$O(m+n)$		$O(m \log(\frac{n}{m}+1))$
Intersection	$O(\min(m,n))$	$O(m \times n)$	$O(m \log(\frac{n}{m}+1))$
Difference	$O(m)$		$O(m \log(\frac{n}{m}+1))$
Insert	$O(1)$	$O(n)$	$O(\log n)$
Delete	$O(1)$	$O(n)$	$O(\log n)$

- Python: Time Complexity
- Haskell: Data.Set and Data.Map.Strict
- Remember that actual behaviour will depend on the data and compiler settings

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Binary Tree Further Exercises

5 Binary Tree Exercises

- Binary Tree shapes
- Generating Binary Trees
- Catalan Numbers (advanced)

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Binary Tree Common Exercises

Interview Questions and Practice Problems

- This section contains some common exercises used in *Google Interview Questions*
- See the References section for Web sites with more examples
- Note that this section is not directly part of M269 and is here for interest and practice using recursion
- Further questions may be added to this section

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Binary Tree Shapes

Activity 30 Shape Exercises

- `isSameShape(t1,t2)` takes two binary trees and returns `True` if they have the same shape
- `isMirrorShape(t1,t2)` takes two binary trees and returns `True` if they are a mirror of each other
- `isSymmetric(t)` takes a binary tree and returns `True` if it is symmetric
- `genMirrorShape(t)` takes a binary tree and returns the mirror of the tree

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Binary Tree Shapes

Answer 30 Shape Exercises — isSameShape(t1,t2)

- isSameShape(t1,t2) takes two binary trees and returns True if they have the same shape

```
206 def isSameShape(t1: ABTree,t2: ABTree) -> bool :
207   if isEmptyABT(t1) and isEmptyABT(t2) :
208     return True
209   elif isEmptyABT(t1) or isEmptyABT(t2) :
210     return False
211   else :
212     (t1L,k1,t1R) = exposeABT(t1)
213     (t2L,k2,t2R) = exposeABT(t2)
214     return (isSameShape(t1L,t2L)
215             and isSameShape(t1R,t2R))
```

- Answer 30 continued on next slide

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Binary Tree Shapes

Answer 30 Shape Exercises — isMirrorShape(t1,t2)

- isMirrorShape(t1,t2) takes two binary trees and returns True if they are a mirror of each other

```
217 def isMirrorShape(t1: ABTree,t2: ABTree) -> bool :
218   if isEmptyABT(t1) and isEmptyABT(t2) :
219     return True
220   elif isEmptyABT(t1) or isEmptyABT(t2) :
221     return False
222   else :
223     (t1L,k1,t1R) = exposeABT(t1)
224     (t2L,k2,t2R) = exposeABT(t2)
225     return (isMirrorShape(t1L,t2R)
226             and isMirrorShape(t1R,t2L))
```

- Answer 30 continued on next slide

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Binary Tree Shapes

Answer 30 Shape Exercises — `isSymmetric(t)`

- `isSymmetric(t)` takes a binary tree and returns `True` if it is symmetric

```
228 def isSymmetric(t: ABTree) -> bool :
229   return isMirrorShape(t,t)
```

- Answer 30 continued on next slide

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Binary Tree Shapes

Answer 30 Shape Exercises — genMirrorShape(t)

- genMirrorShape(t) takes a binary tree and returns the mirror of the tree

```
231 def genMirrorShape(t: ABTree) -> ABTree :
232   if isEmptyABT(t) :
233     return mkEmptyABT()
234   else :
235     (tL,x,tR) = exposeABT(t)
236     return (mkNodeABT(x, genMirrorShape(tR),
237                          genMirrorShape(tL)))
```

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Binary Tree Shapes

Isomorphic Binary Trees (a)

- Two binary trees are isomorphic if one can be obtained from the other by flipping the left and right subtrees. Two empty trees are isomorphic
- See Tree isomorphism problem
- See Is the recursive approach to binary tree isomorphism actually linear?

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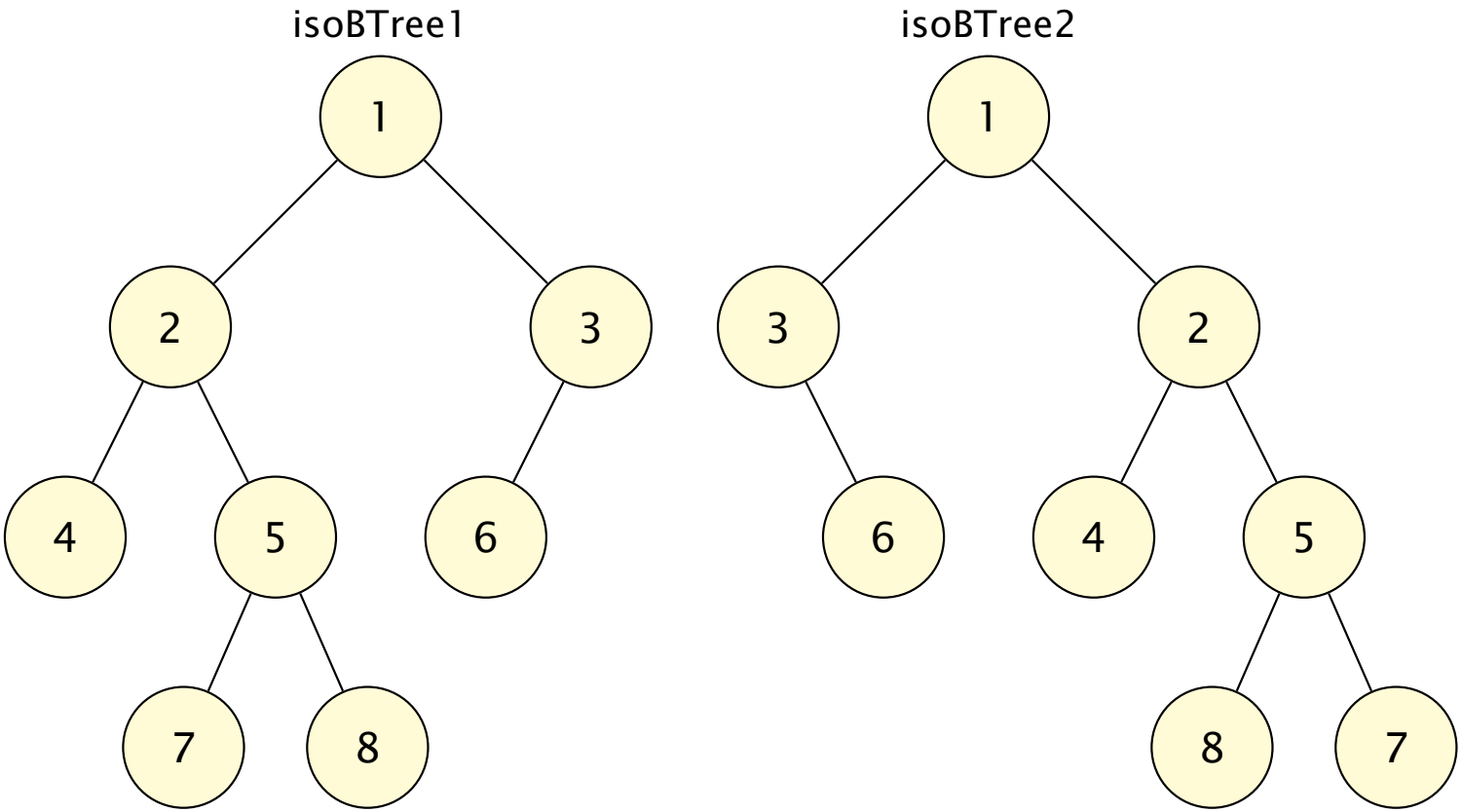
Binary Trees

Isomorphic Binary Trees (b) `isoBTree1`, `isoBTree2`



- `isoBTree1`, `isoBTree2` are isomorphic with the following flips:

  (2,3), (`EmptyBTree`, 6), (7,8)

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Binary Tree Shapes

Isomorphic Binary Trees (c) Python Code

```
def isIsomorphic(t1 : ABTree, t2 : ABTree) - > bool :
  if isEmptyABT(t1) and isEmptyABT(t2) :
    return True
  elif isEmptyABT(t1) or isEmptyABT(t2) :
    return False
  else :
    (t1L,k1,t1R) = exposeABT(t1)
    (t2L,k2,t2R) = exposeABT(t2)
    if (k1 != k2) :
      return False
    else :
      return ((isIsomorphic(t1L, t2L) and isIsomorphic(t1R,t2R))
              or
              (isIsomorphic(t1L, t2R) and isIsomorphic(t1R,t2L))
             )
```


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Generating Binary Trees

Exercises

- The aim is to generate the shapes of all possible trees given a number of nodes
- First sketch a few trees to spot any pattern
- Write down a recurrence relation for the number of binary tree shapes with n nodes based on the number of tree shapes for less than n nodes
- Write a function `genBTs(x,n)` given a value `x` and an integer `n` generates the Python representation of all shapes of binary trees with `n` nodes with `x` at each node

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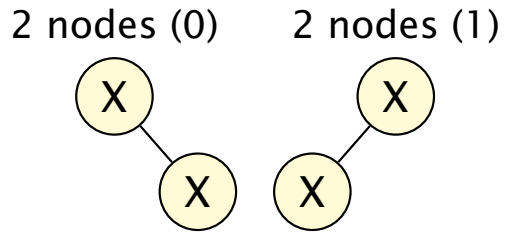
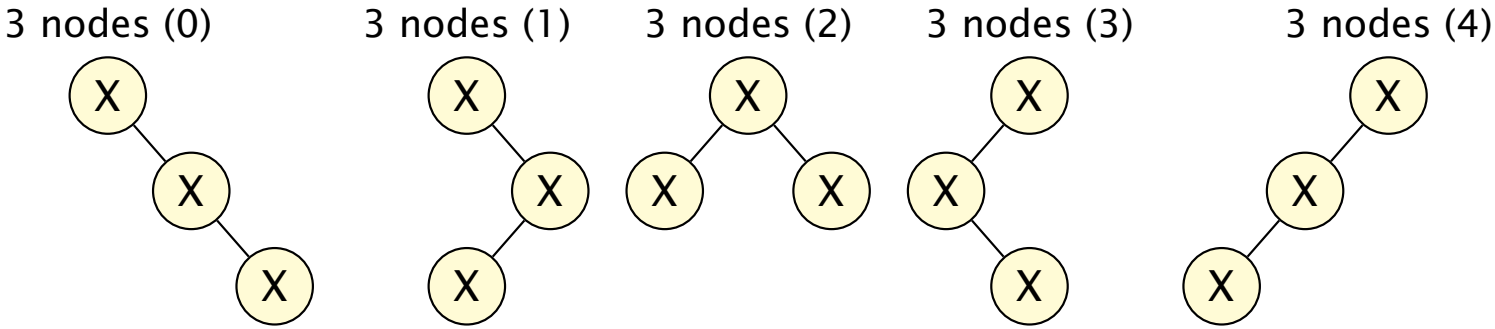
Generating Binary Trees

Simple Recursive Version (1)

- We first sketch a few trees to spot the pattern
- 0 nodes have 1 tree, `EmptyBT`, 1 node has 1 tree
- 2 nodes have 2 trees



- 3 nodes have 5 trees



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Generating Binary Trees

Simple Recursive Version (2)

- Let C_n be the number of binary tree shapes with n nodes then from the above diagrams we have:
- $C_0 = 1$
- $C_1 = 1$
- $C_2 = 2$
- $C_3 = 5$
- Eureka insight for a tree with n nodes if the left subtree has i nodes then the right subtree must have $n - i - 1$ nodes and i can range over 0 to $n - 1$
- The left and right subtrees must have C_i and C_{n-i-1} different possible shapes
- and there are n possible values for i from 0 to $n - 1$
- Hence $C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$

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Generating Binary Trees

Simple Recursive Version (3)

- $C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1} = \sum_{i=1}^{n} C_{i-1} C_{n-i}$
- Alternatively $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$
- Check $C_1 = C_0 C_0 = 1 \times 1 = 1$
- $C_2 = C_0 C_1 + C_1 C_0 = 1 \times 1 + 1 \times 1 = 2$
- $C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0 = 1 \times 2 + 1 \times 1 + 2 \times 1 = 5$
- $C_4 = C_0 C_3 + C_1 C_2 + C_2 C_1 + C_3 C_0$
  $= 1 \times 5 + 1 \times 2 + 2 \times 1 + 5 \times 1 = 14$
- The C_n are known as the Catalan numbers

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Simple Recursive Version (4)

- The simple recursive definition of genBTs follows from the recurrence relation directly
- Uses Python List Comprehensions see below
- This repeats the calculation of subtrees
- This is similar to the definition in Math.Combinat.Trees.Binary which is based on Knuth (2011, section 7.2.1.6), Knuth (1997, section 2.3.4.4)

```
239 def genABTs(x: T,n: int) -> [ABTree] :
240   if n == 0 :
241     return [mkEmptyABT()]
242   elif n == 1 :
243     return [mkNodeABT(x,mkEmptyABT(),mkEmptyABT())]
244   else :
245     ts = ([mkNodeABT(x,leftT,rightT)
246            for (nu,nv) in splitsInt(n)
247            for leftT   in genABTs(x,nu)
248            for rightT  in genABTs(x,nv)])
249     return ts

251 def splitsInt(n: int) -> [(int,int)] :
252   prns = [(i, n - i - 1) for i in range(n)]
253   return prns
```


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List Comprehensions

- List comprehensions (tutorial), List comprehensions (reference) — a neat way of expressing iterations over a list, came from Miranda (see Wikipedia: List comprehension)
- Example: Square the even numbers between 0 and 9

```
Python3>>> [x ** 2 for x in range(10) if x % 2 == 0]
[0, 4, 16, 36, 64]
Python3>>> [(x,y) for x in range(4)
...                for y in range(4)
...                if x % 2 == 0
...                   and y % 3 == 0]
[(0, 0), (0, 3), (2, 0), (2, 3)]
Python3>>>
```

- In general

```
  [expr for target1 in iterable1 if cond1
        for target2 in iterable2 if cond2 ...
        for targetN in iterableN if condN ]
```


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Catalan Numbers

Efficient Calculation

- As with many other problems, it may be easier to find a recursive relation or recurrence for a problem and harder to find an efficient calculation.
- For the Catalan numbers it is possible to find a closed (non-recursive) expression for the Catalan numbers
- Below is a derivation of a closed expression — this is not part of M269 and is included for interest — the derivation uses a bit more Maths than the rest of these notes but it is explained as we progress
- This derivation is from Spivey (2019, page 208) *The Art of Proving Binomial Identities* and Wilf (1994, page 44) *generatingfunctionology*

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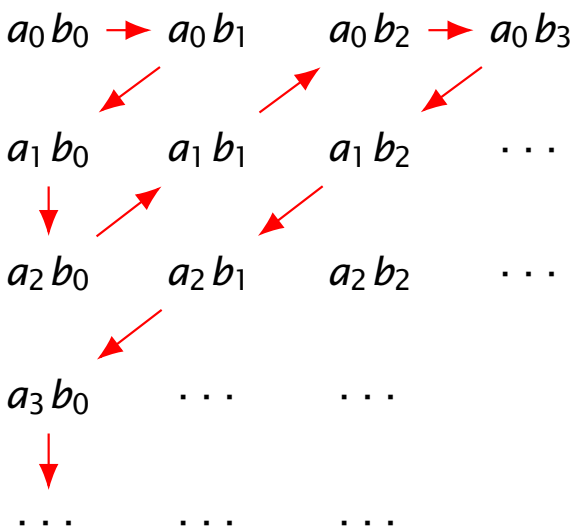
Cauchy Product

Products of Infinite Series or Power Series (1)

- $$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{n=0}^{\infty} c_n x^n$$

  where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

- The product forms a two-dimensional array — however we can arrange a sequence that goes through the array — see below and Spivak (2008, p486, p493, p513)



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Cauchy Product

Products of Infinite Series or Power Series (2)

- $$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right)$$
  $$= (a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$$
  $$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$
- See Wikipedia: Cauchy product
- If we have $c(x) = \sum_{i=0}^{\infty} c_i x^i$
- $$(c(x))^2 = \left(\sum_{i=0}^{\infty} c_i x^i\right)\left(\sum_{j=0}^{\infty} c_j x^j\right)$$
  $$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} c_k c_{n-k}\right) x^n$$
- This result is used in finding a closed form for the Catalan numbers
- Based on Mike Spivey 2013

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Catalan Numbers

Catalan Recurrence (1)

- $C_0 = 1$
- $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$
- Define $c(x)$ to be the generating function of the infinite sequence of the Catalan numbers
- $c(x) = \sum_{n=0}^{\infty} C_n x^n$
- Hence we can multiply both sides of the recurrence by x^n and sum
- $\sum_{n=0}^{\infty} C_{n+1} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n$

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Catalan Recurrence (2)

- $\sum_{n=0}^{\infty} C_{n+1} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n$
- $\frac{1}{x} \sum_{n=0}^{\infty} C_{n+1} x^{n+1} = (c(x))^2$ by Cauchy product
- $\frac{1}{x} \left(\sum_{n=0}^{\infty} C_n x^n - C_0 \right) = (c(x))^2$
- $\frac{1}{x}(c(x) - 1) = (c(x))^2$
- $x(c(x))^2 - c(x) + 1 = 0$
- $c(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}$
- We know $c(0) = C_0 = 1$

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Catalan Recurrence (3)

- $c(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}$
- We know $c(0) = C_0 = 1$
- Applying L'Hôpital's rule

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

- $\lim_{x \to 0^+} \frac{1 - \sqrt{1 - 4x}}{2x} = \lim_{x \to 0^+} \frac{2(1 - 4x)^{-\frac{1}{2}}}{2} = 1$
- Hence $c(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$
- We now use the generalised Binomial theorem to expand this expression

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Catalan Recurrence (4)

- The generalised Binomial theorem has

  If $|x| > |y|$ and r is any complex number then

  $$(x+y)^r = \sum_{k-0}^{\infty} \binom{r}{k} x^{r-k} y^k$$

  where $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$

- $c(x) = \frac{1}{2x}(1-\sqrt{1-4x}) = \frac{1}{2x}\left(1 - \sum_{n=0}^{\infty} \binom{1/2}{n}(-4x)^n\right)$

- The coefficient of x^n expands to

  $$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}(-4)^n$$

  $$= \frac{1(1-2)\cdots(1-2n+2)}{n!}(-1)^n 2^n$$

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Catalan Recurrence (5)

- The coefficient of x^n expands to

$$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}(-4)^n$$

$$= \frac{1(1-2)\cdots(1-2n+2)}{n!}(-1)^n 2^n$$

$$= \frac{(1)(3)\cdots(2n-3)(-1)^{n-1}}{(n!)^2}(-1)^n 2^n (n!)$$

$$= \frac{(1)(3)\cdots(2n-3)(-1)^{n-1}}{(n!)^2}(-1)^n (2n)(2n-2)\cdots(2)$$

$$= -\frac{(2n)!}{(n!)^2(2n-1)}$$

$$= -\binom{2n}{n}\frac{1}{2n-1}$$

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Catalan Recurrence (6)

- Hence $c(x) = \frac{1}{2x}\left(1 + \sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2n-1}x^n\right)$

$$= \frac{1}{2x}\left(1 + (-1) + \sum_{n=1}^{\infty}\binom{2n}{n}\frac{1}{2n-1}x^n\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{1}{2n-1}x^{n-1}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\binom{2(n+1)}{n+1}\frac{1}{2n+1}x^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(2n+2)(2n+1)}{(n+1)^2}\binom{2n}{n}\frac{1}{2n+1}x^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$$

- Hence $C_n = \frac{1}{n+1}\binom{2n}{n}$

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Sample Catalan Numbers

Mathematica code

```
  In[1]:= Series[(1 - Sqrt[1-4x])/(2x),{x,0,12}]
  Out[1]= SeriesData[x, 0, {1, 1, 2, 5, 14, 42, 132, 429, 1430, \
4862, 16796, 58786, 208012}, 0, 13, 1]
```

- Generating function form
- $1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + 1430x^8 + 4862x^9 + 16796x^{10} + 58786x^{11} + 208012x^{12} + O\left(x^{13}\right)$

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6 Tutorial End, References and Colophon

- Future work and dates
- References to other Python texts or documentation
- References to other computing material
- Article version has the full references and bibliography with back references
- **Colophon**
- LaTeX with Beamer, Listings and other packages
- Index of Python code and diagrams
- PGF/TikZ for the diagrams
- External copies of the diagrams as PDF with tight bounding boxes are available

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Future Work

Graph algorithms, Greed, Logic, Computability

- Hashing and hash tables
- Binary search trees, height balanced binary search trees, AVL trees
- Graph algorithms
- Greedy algorithms
- Logic, Computability
- Future dates for tutorials and TMAs

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Python

Web Links & References

- **Lutz (2013)** *Learning Python* — one of the best introductory books
- **Lutz (2011)** *Programming Python* — a more advanced book
- **Martelli et al (2023)** *Python in a Nutshell*
- **Ramalho (2022)** *Fluent Python* a more advanced book
- **Python 3 Documentation**
  https://docs.python.org/3/
- **Python Style Guide PEP 8**
  https://www.python.org/dev/peps/pep-0008/
  (Python Enhancement Proposals)

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Haskell

Web Links & References

- **Haskell Language** https://www.haskell.org
- **HaskellWiki** https://wiki.haskell.org/Haskell
- **Learn You a Haskell for Great Good!** http://learnyouahaskell.com — very readable introduction to Haskell
- **Bird and Wadler (1988); Bird (1998, 2014)** — one of the best introductions but tough in parts, requires some mathematical maturity — the three books are in effect different editions
- Bird, Gibbons (2020) *Algorithm Design with Haskell* — developing the algorithms in a purely functional way
- **Functors, Applicatives, and Monads in Pictures** http://adit.io/posts/2013-04-17-functors,_applicatives,_and_monads_in_pictures.html — a very good outline with cartoons
- **Haskell Wikibook** https://en.wikibooks.org/wiki/Haskell

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