M269 Units 3,4,5

Sorting, Searching, Graph Algorithms

Contents

1 Agenda

- Welcome and introductions
- Session on M269 *Sorting*, *Abstract Data Types*, *Searching*, *Graph Algorithms*
- Recursion
- String search
- Bags and Abstract Data Types (and Objects)
- Search Trees and Height Balanced Trees

- Graph Algorithms, Greedy algorithm example — Interval Scheduling
- Dynamic Programming — Edit Distance example
- Implementations in Structured English, Python and Haskell (Optional)
- *Note* there is more material here than we can cover — some is for optional interest
- Slides, notes and source code files are at
- M269 Tutorial Sorting
- M269 Tutorial Binary Trees
- M269 Tutorial Graphs
- M269 Tutorial Units 3,4,5

Introductions — Me

- *Name* Phil Molyneux
- *Background* Physics & Maths, Operational Research, Computer Science
- *First programming languages* Fortran, BASIC, Pascal
- *Favourite Software*
  - Haskell — pure functional programming language
  - Text editors TextMate, Sublime Text — previously Emacs
  - Word processing in LaTeX — all these slides and notes
  - Mac OS X
- *Learning style* — I read the manual before using the software

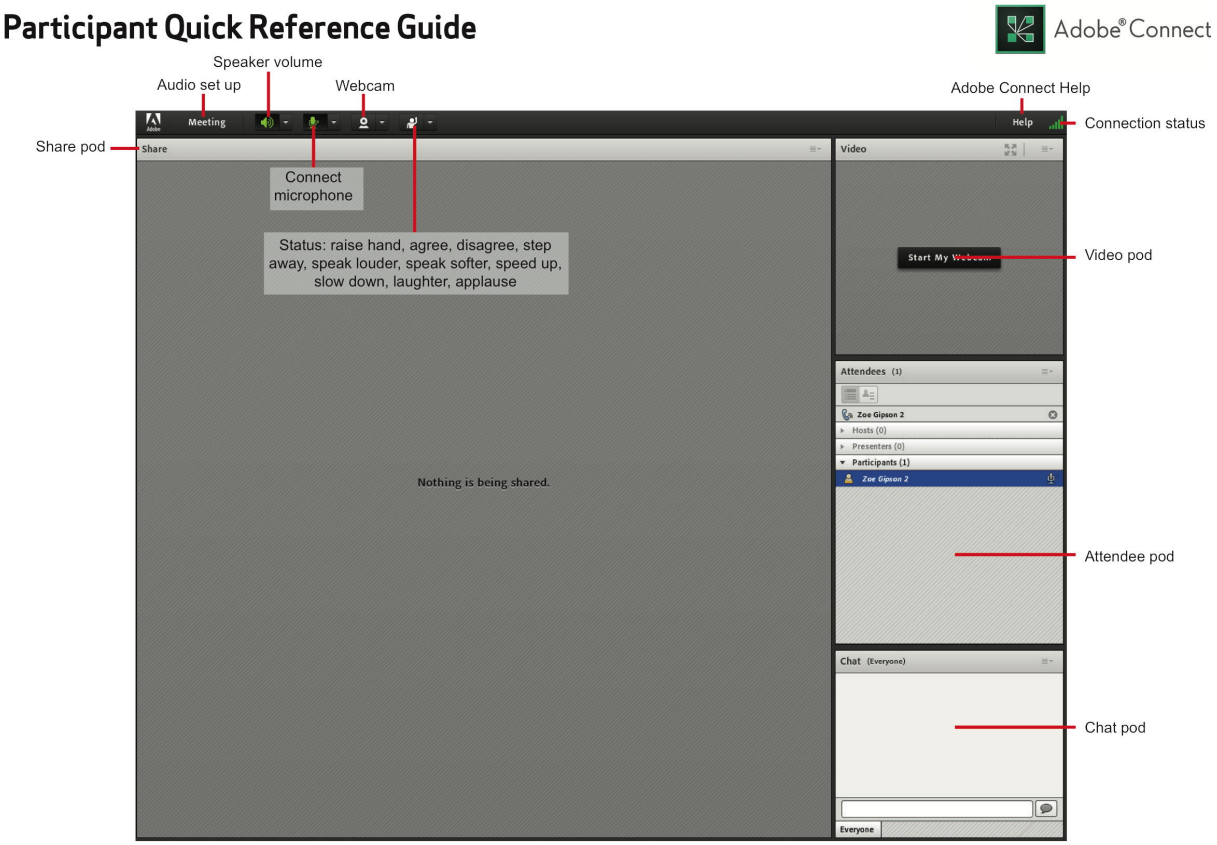
Introductions — You

- *Name* ?
- *New topics last month* ?
- *M269 Unit 3,4,5 Sorting, Searching, Graph Algorithm topics you want covered* ?
- *Learning style* ?
- *Other OU courses* ?
- *Anything else* ?

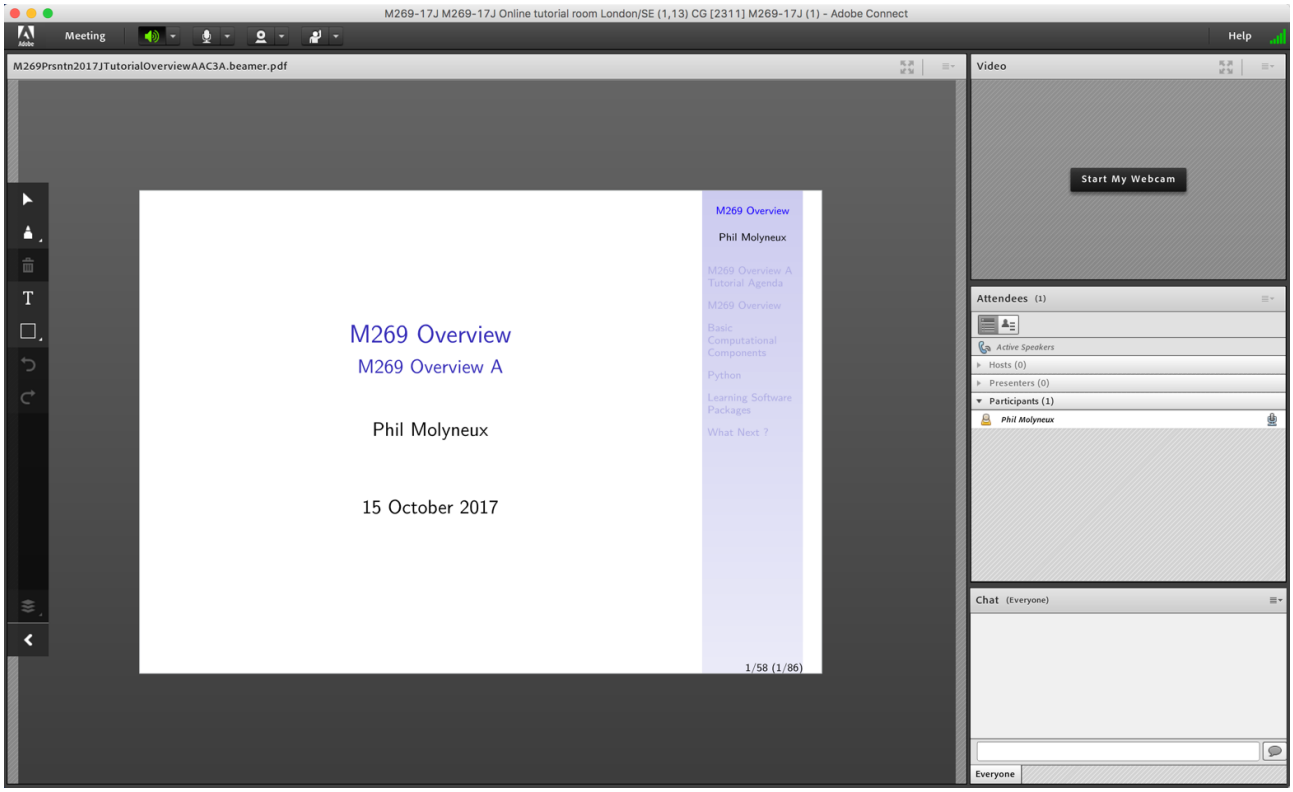
2 Adobe Connect Interface and Settings

2.1 Adobe Connect Interface — Student View

Adobe Connect Interface — Student Quick Reference



Adobe Connect Interface — Student View



2.2 Adobe Connect Settings

Adobe Connect Settings

- **Everybody: Audio Settings** Meeting › Audio Setup Wizard...
- **Audio** Menu bar › Audio › Microphone rights for Participants ✔
- Do not *Enable single speaker mode*
- **Drawing Tools** Share pod menu bar › Draw (1 slide/screen)
- Share pod menu bar › Menu icon › Enable Participants to draw ✔ gray
- Meeting › Preferences › Whiteboard › Enable Participants to draw ✔
- Cancel hand tool ... Do not enable green pointer...
- Meeting › Preferences › Attendees Pod ✘ *Raise Hand notification*
- Meeting › Preferences › Display Name Display First & Last Name
- **Cursor** Meeting › Preferences › General tab › Host Cursors › Show to all attendees ✔ (default *Off*)
- Meeting › Preferences › Screen Share › Cursor › Show Application Cursor
- **Webcam** Menu bar › Webcam › Enable Webcam for Participants ✔
- **Recording** Meeting › Record Meeting... ✔

Adobe Connect — Access

- **Tutor Access**

  TutorHome › M269 Website › Tutorials

  Cluster Tutorials › M269 Online tutorial room

  Tutor Groups › M269 Online tutor group room

  Module-wide Tutorials › M269 Online module-wide room

- **Attendance**

  TutorHome › Students › View your tutorial timetables

- **Beamer Slide Scaling** 440% (422 x 563 mm)
- **Clear Everyone's Status**

  Attendee Pod › Menu › Clear Everyone's Status

- **Grant Access** and send link via email

  Meeting › Manage Access & Entry › Invite Participants...

- **Presenter Only Area**

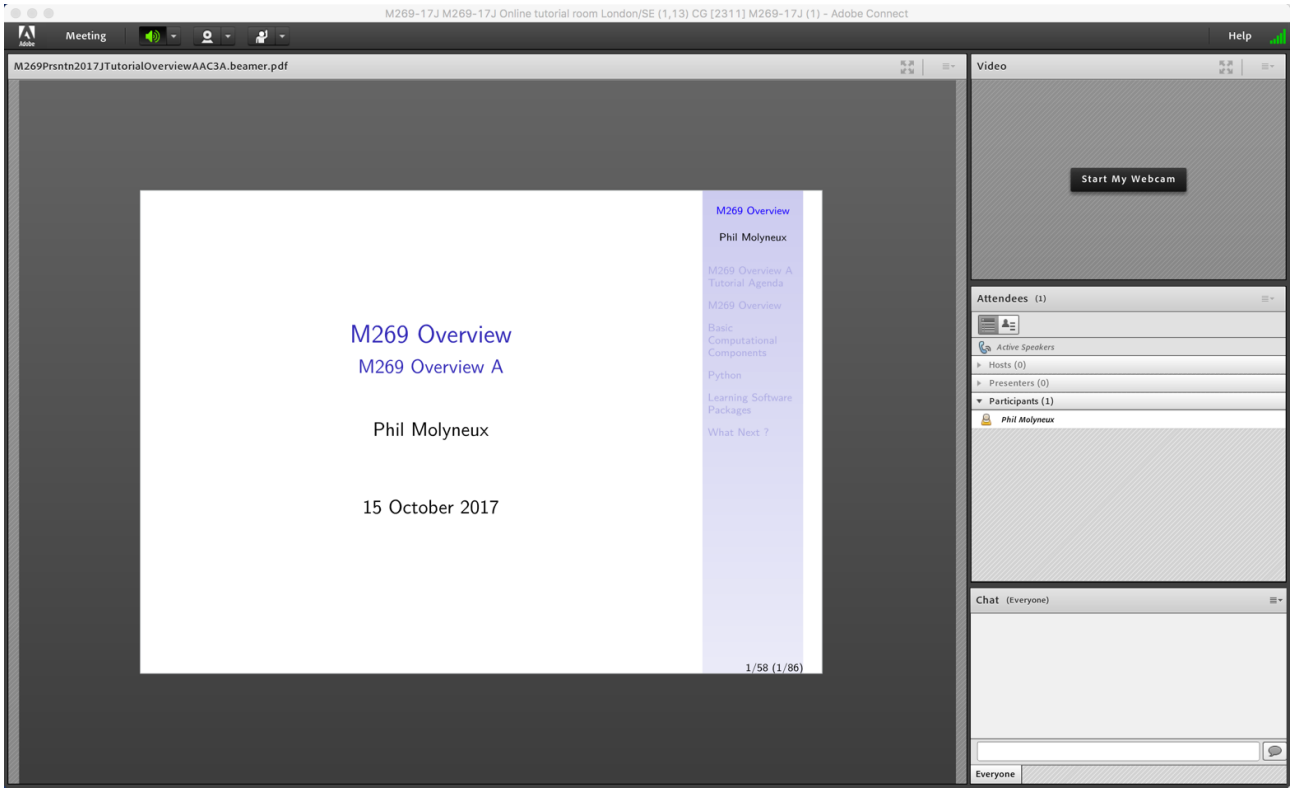
  Meeting › Enable/Disable Presenter Only Area

Adobe Connect — Keystroke Shortcuts

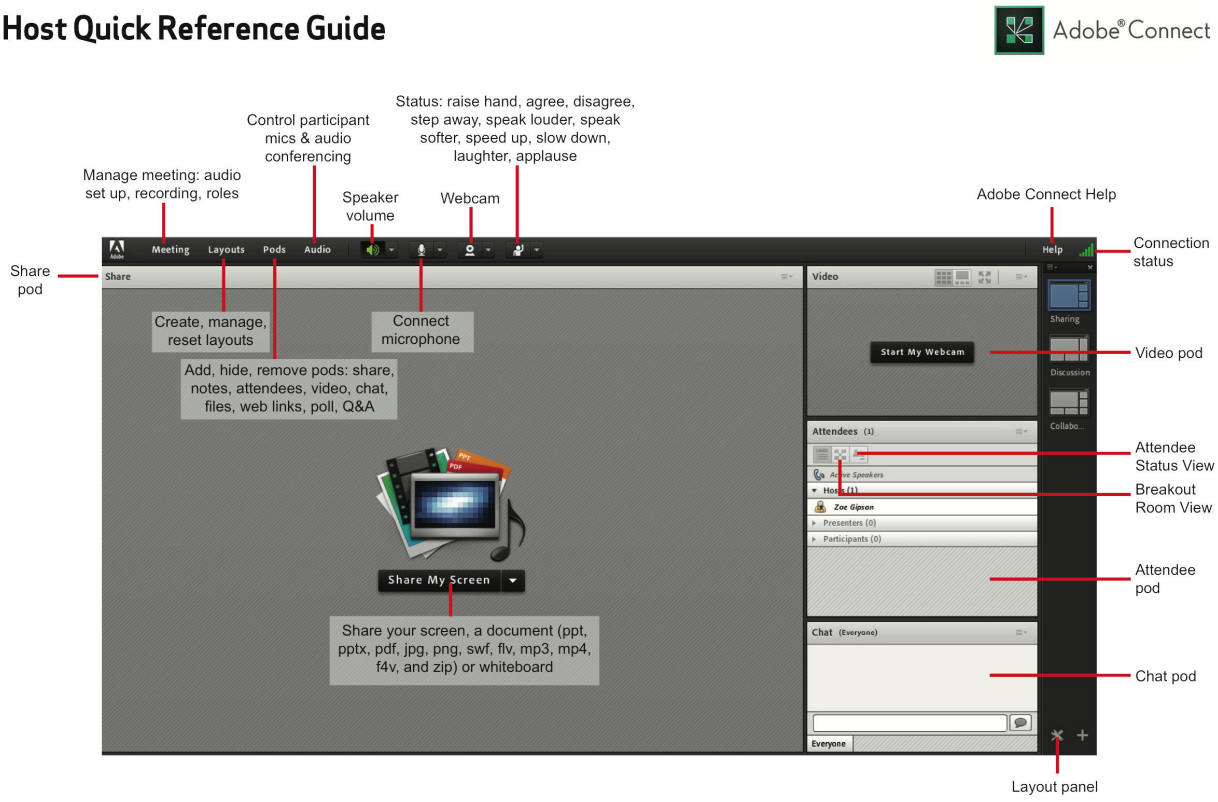
- Keyboard shortcuts in Adobe Connect
- **Toggle Mic** ⌘ + M (Mac), Ctrl + M (Win) (On/Disconnect)
- **Toggle Raise-Hand status** ⌘ + E
- **Close dialog box** ⎋ (Mac), Esc (Win)
- **End meeting** ⌘ + \

2.3 Adobe Connect Interface — Student & Tutor Views

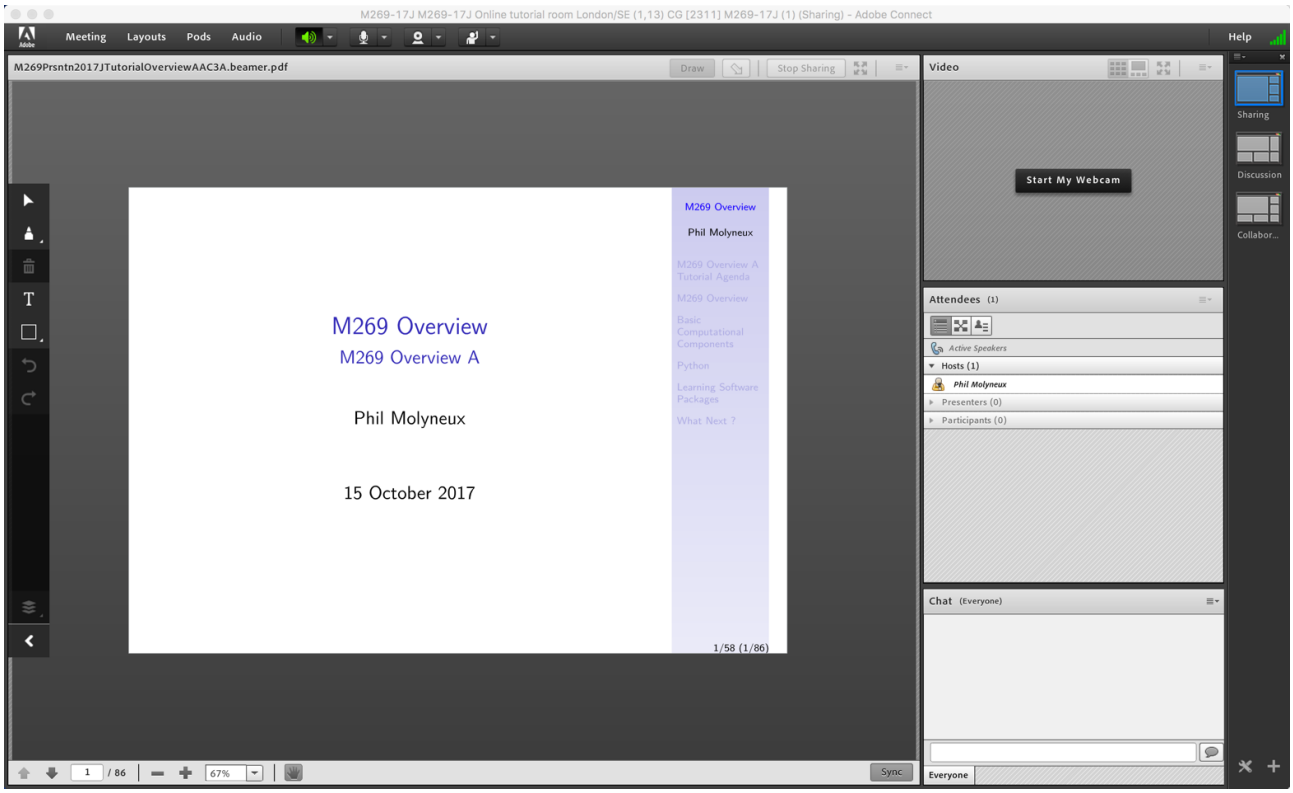
Adobe Connect Interface — Student View (default)



Adobe Connect Interface — Tutor Quick Reference



Adobe Connect Interface — Tutor View



2.4 Adobe Connect — Sharing Screen & Applications

- Share My Screen › Application tab › Terminal for Terminal
- Share menu › Change View › Zoom in for mismatch of screen size/resolution (Participants)
- (Presenter) Change to 75% and back to 100% (solves participants with smaller screen image overlap)
- Leave the application on the original display
- Beware blued hatched rectangles — from other (hidden) windows or contextual menus
- Presenter screen pointer affects viewer display — beware of moving the pointer away from the application
- First time: System Preferences › Security & Privacy › Privacy › Accessibility

2.5 Adobe Connect — Ending a Meeting

- *Notes for the tutor only*
- **Student:** Meeting › Exit Adobe Connect
- **Tutor:**
- **Recording** Meeting › Stop Recording ✔
- **Remove Participants** Meeting › End Meeting... ✔
  - Dialog box allows for message with default message:
  - *The host has ended this meeting. Thank you for attending.*
- **Recording availability** *In course Web site for joining the room, click on the eye icon in the list of recordings under your recording* — edit description and name
- **Meeting Information** Meeting › Manage Meeting Information — can access a range of information in Web page.
- **Attendance Report** see course Web site for joining room

2.6 Adobe Connect — Invite Attendees

- **Provide Meeting URL** Menu › Meeting › Manage Access & Entry › Invite Participants...
- **Allow Access without Dialog** Menu › Meeting › Manage Meeting Information provides new browser window with *Meeting Information* Tab bar › Edit Information
- Check *Anyone who has the URL for the meeting can enter the room*
- Default *Only registered users and accepted guests may enter the room*
- **Reverts to default next session but URL is fixed**
- Guests have blue icon top, registered participants have yellow icon top — same icon if URL is open

- See Start, attend, and manage Adobe Connect meetings and sessions

2.7 Layouts

- **Creating new layouts** example *Sharing* layout
- Menu > Layouts > Create New Layout... > Create a New Layout dialog > Create a new blank layout and name it *PMolyMain*
- New layout has no Pods but does have Layouts Bar open (see Layouts menu)
- **Pods**
- Menu > Pods > Share > Add New Share and resize/position — initial name is *Share n*
- **Rename Pod** Menu > Pods > Manage Pods... > Manage Pods > Select > Rename or Double-click & rename
- Add Video pod and resize/reposition
- Add Attendance pod and resize/reposition
- Add Chat pod — name it *PMolyChat* — and resize/reposition
- Dimensions of **Sharing** layout (on 27-inch iMac)
  - Width of Video, Attendees, Chat column 14 cm
  - Height of Video pod 9 cm
  - Height of Attendees pod 12 cm
  - Height of Chat pod 8 cm
- **Duplicating Layouts** does *not* give new instances of the Pods and is probably not a good idea (apart from local use to avoid delay in reloading Pods)

2.8 Chat Pods

- **Format Chat text**
- Chat Pod > menu icon > My Chat Color
- Choices: Red, Orange, Green, Brown, Purple, Pink, Blue, Black
- Note: Color reverts to Black if you switch layouts
- Chat Pod > menu icon > Show Timestamps

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3 Recursion

3.1 Recursion Examples — Introduction

- These notes give several examples of recursive programs similar to the Python Activities in M269 Unit 3

- Each example gives a function definition for the stated problem followed by a sample evaluation of the function applied to a small argument
- This is followed by a version of the program code similar to that of Python Activities 3.2 to 3.5 with print statements giving the value of some variables on call and return of the function
- Since print statements embedded in a recursive function can be as hard to follow as the recursion itself, there are two further implementations with pure functions that accumulate the call and return messages and also a version which returns the evaluation messages for a sample evaluation — this aims to give a further perspective on thinking about recursion

3.2 Factorial Example

3.2.1 Factorial Definition

- `factorial()` is a standard introduction to recursion
- `factorial(n) = n * (n-1) * ... * 2 * 1`

```
10 def factorial(n) :
11   if (n == 1) :              # line (A)
12     return 1
13   else :                     # line (B)
14     return (n * factorial(n - 1))
```

- Below is a sample evaluation of `factorial(6)`
- To evaluate a function applied to actual arguments, we evaluate the body of the function with each formal parameter replaced by the corresponding actual argument
- In the evaluation below we refer to line (A) or `11` and line (B) or `13` (of file `Recursion2020PrintC`

```
# factorial(6) evaluation
Call 1 -> 6 * factorial(5)
                    # by line (B)
Call 2 -> 6 * (5 * factorial(4))
                    # by line (B)
Call 3 -> 6 * (5 * (4 * factorial(3)))
                    # by line (B)
Call 4 -> 6 * (5 * (4 * (3 * factorial(2))))
                    # by line (B)
Call 5 -> 6 * (5 * (4 * (3 * (2 * factorial(1)))))
                    # by line (B)
Call 6 -> 6 * (5 * (4 * (3 * (2 * (1 * 1)))))
                    # by line (A)
factorial(6) = 720
```

- In writing down an evaluation by hand it is important to hang on to the brackets — they indicate a subexpression that has to be evaluated before an outer expression
- It is possible to write the definition differently to accumulate the product as the computation goes — see later

3.2.2 Factorial with Print Statements

```
18 def facWithPrint(n) :
19   return helpFacWithPrint(n, 1, 1)

21 def helpFacWithPrint(n, call, prodAccum) :
22   if (n == 1) :              # line (A)
23     callMsg = calcFacCallMsg(call, n)
24     print(callMsg)
25     retMsg  = calcFacRetMsg(call, n, prodAccum)
26     print(retMsg)
27     return (n * prodAccum)
28   else :                     # line (B)
29     callMsg = calcFacCallMsg(call, n)
30     print(callMsg)
31     pAccum = helpFacWithPrint(n-1, call+1
32                               ,prodAccum)
33     retMsg  = calcFacRetMsg(call, n, pAccum)
34     print(retMsg)
35     return (n * pAccum)
```

- Below is the output from `facWithPrint(6)` and some comments on the code

```
Python3>>> facWithPrint(6)
Call 1 of function with n = 6
Call 2 of function with n = 5
Call 3 of function with n = 4
Call 4 of function with n = 3
Call 5 of function with n = 2
Call 6 of function with n = 1
Return call 6 returning 1 * 1
Return call 5 returning 2 * 1
Return call 4 returning 3 * 2
Return call 3 returning 4 * 6
Return call 2 returning 5 * 24
Return call 1 returning 6 * 120
720
Python3>>>
```

- Here `Python3>>>` is my Python shell prompt
- Below are some comments on the code
- `facWithPrint()`, `helpFacWithPrint()` description
- `facWithPrint()` calls `helpFacWithPrint(n, 1, 1)`
- The arguments to `helpFacWithPrint()` are:
- `n` initialised to the argument of `facWithPrint()`
- `call` the call number initialised to `1`
- `prodAccum` is the base case (initialised to `1`) — this argument is not needed but there may be generalisations
- The recursive call to `helpFacWithPrint()` at line `31` increments `call` — this mimics a global variable
- The calls to `calcFacCallMsg()`, `calcFacRetMsg()` at lines `23`,`29`,`25`,`33` construct the messages which are then printed — we could have lifted the two calls to `calcFacCallMsg` to just one before the `if`

3.2.3 Factorial with Call, Return Messages

- The snag with inserting print statements into code is the program ceases to be a pure function — rather it is a program that takes a string and returns a computation

which does some I/O and returns the factorial

- This conflates understanding the original pure function with understanding where the output comes from in code with side effects
- This can be written as a pure function which takes a number and returns a triple of the output number, and lists of call and return messages, and then decide how to present this

```
41 def funcFacWithMsgs(n) :
42   return helpFuncFacWM(n, 1, 1, [], [])
```

```
44 def helpFuncFacWM(n, call, prodAccum
45                   , callMsgList, retMsgList) :
46   callMsg = calcFacCallMsg(call, n)
47   if (n == 1) :               # line (A)
48     retMsg  = calcFacRetMsg(call, n, prodAccum)
49     return (n * prodAccum
50             , [callMsg] + callMsgList
51             , [retMsg] + retMsgList )
52   else :                      # line (B)
53     (pAccum, pCallMsgList, pRetMsgList) \
54       = helpFuncFacWM(n-1, call+1, prodAccum
55                       , callMsgList, retMsgList)
56     retMsg  = calcFacRetMsg(call, n, pAccum)
57     return (n * pAccum
58             , [callMsg] + pCallMsgList
59             , [retMsg] + pRetMsgList )
```

- `helpFuncFacWM()` description — the arguments are:
- `n` initialised to argument of `funcFacWithMsgs()`
- `call` the call number initialised to `1`
- `prodAccum` is the base case (initialised to `1`) — this argument is not needed but there may be generalisations
- `callMsgList`, the call messages list, initialised to an empty list
- `retMsgList`, the return messages list, initialised to an empty list
- The recursive call to `helpFuncFacWM()` at line `54` increments `call` — mimics a global variable
- The call to `calcFacCallMsg()` is now before the `if` at line `46` — constructs the call message
- The calls to `calcFacRetMsg()` at lines `48`,`56` construct the return message

```
61 def calcFacCallMsg(call, n) :
62   return ("Call␣" + str(call)
63           + "␣of␣function␣with␣n␣=␣" + str(n) )

65 def calcFacRetMsg(call, n, acc) :
66   return ("Return␣call␣" + str(call)
67           + "␣returning␣" + str(n)
68           + "␣*␣" + str(acc) )
```

```
72 nl = "\n"

74 sp = "␣"
75 sp20 = sp.join(['']*21) # string of 20 spaces

77 def pStr(aStr) :
78   return ("\"" + aStr + "\"")

80 def printFacWM(n) :
81   (facFinal, callMsgList, retMsgList) \
```

```
82      = funcFacWithMsgs(n)
83    print(nl.join(callMsgList)
84      + nl
85      + nl.join(reversed(retMsgList))
86      + nl + "factorial("
87      + str(n) + ")␣=␣"
88      + str(facFinal) )
```

- `nl`, `sp` are defined to make the code easier to read
- `join()` and `reversed()` are Python built-in functions
- The return messages list is reversed to get the messages in the same order as the print statement version — this might take some thinking
- `pStr()` is needed since `str("")` does not produce `'""'`
- Sample output from `printFacWM(6)`

```
Python3>>> printFacWM(6)
Call 1 of function with n = 6
Call 2 of function with n = 5
Call 3 of function with n = 4
Call 4 of function with n = 3
Call 5 of function with n = 2
Call 6 of function with n = 1
Return call 6 returning 1 * 1
Return call 5 returning 2 * 1
Return call 4 returning 3 * 2
Return call 3 returning 4 * 6
Return call 2 returning 5 * 24
Return call 1 returning 6 * 120
factorial(6) = 720
Python3>>>
```

3.2.4 Factorial with Evaluation Messages

- The evaluation messages can also be generated
- The code requires a bit more thought since the evaluation messages are also recursive

```
92  def funcFacEval(n) :
93    return helpFuncFacEval(n, 1, 1, [], [])
```

```
95  def helpFuncFacEval(n, call, prodAccum
96                      , argList, evalMsgList) :
97    if (n == 1) :               # line (A)
98      reasonStr = "#␣by␣line␣(A)"
99      evalStr = str(n)
100     evalMsg  = calcFacEvalMsg(call, n
101                               , argList, evalStr
102                               , reasonStr)
103     return (n * prodAccum
104           , [n] + argList, [evalMsg] + evalMsgList )
105   else :                      # line (B)
106     reasonStr = "#␣by␣line␣(B)"
107     evalStr = "factorial(" + str(n-1) + ")"
108     evalMsg  = calcFacEvalMsg(call, n
109                               , argList, evalStr
110                               , reasonStr)
111     (pAccum, pArgList, pEvalMsgList) \
112       = helpFuncFacEval(n-1, call+1
113               , prodAccum, [n] + argList, evalMsgList)
114     return (n * pAccum
115           , pArgList, [evalMsg] + pEvalMsgList )
```

- helpFuncFacEval() description — the arguments are:
- n initialised to the argument of funcFacEval()
- call the call number initialised to 1
- prodAccum is the base case (initialised to 1) — this argument is not needed but there may be generalisations
- argList, the previous arguments list, initialised to an empty list — the previous arguments are needed to calculate the evaluation message
- evalMsgList, the evaluation messages list, initialised to an empty list
- The recursive call to helpFuncFacEval() at line 112 increments call — this mimics a global variable
- The calls to calcFacEvalMsg() at lines 100,108 construct the evaluation message (see description below)

```
117 def calcFacEvalMsg(call, n
118                   , argList, evalStr
119                   , reasonStr) :
120   if (argList == []) :
121     retMsg = ("Call␣" + str(call)
122               + "␣->␣" + str(n)
123               + "␣*␣" + evalStr
124               + nl + sp20 + reasonStr)
125     return retMsg
126   else :
127     nextN = argList[0]
128     nextArgList = argList[1:]
129     nextEvalStr \
130       = "(" + str(n) + "␣*␣" + evalStr + ")"
131     return calcFacEvalMsg(call, nextN
132                           , nextArgList, nextEvalStr
133                           , reasonStr)
```

- Notice calcFacEvalMsg() is recursive unlike calcFacCallMsg(), calcFacRetMsg()
- calcFacEvalMsg() description — the arguments are:
- call the current call number
- n the current argument
- argList the list of previous arguments
- evalStr the expression on the right hand side of the current operation (see print output below)
- reasonStr is the string identifying the branch of the if used
- If argList is empty, a string is returned
- Otherwise there is a recursive call to wrap the constructed string with the outer arguments (see print output below)
- It is easy to get the arguments in the wrong order — perhaps this is why this function is not given often.

```
145 def printFacEval(n) :
146   (facFinal, argList, evalMsgList) \
147     = funcFacEval(n)
148   print(nl.join(evalMsgList)
149     + nl + "factorial("
150     + str(n) + ")␣=␣"
151     + str(facFinal) )
```

- Sample output from printFacEval(6)

```
Python3>>> printFacEval(6)
Call 1 -> 6 * factorial(5)
                    # by line (B)
Call 2 -> 6 * (5 * factorial(4))
                    # by line (B)
Call 3 -> 6 * (5 * (4 * factorial(3)))
                    # by line (B)
Call 4 -> 6 * (5 * (4 * (3 * factorial(2))))
                    # by line (B)
Call 5 -> 6 * (5 * (4 * (3 * (2 * factorial(1)))))
                    # by line (B)
Call 6 -> 6 * (5 * (4 * (3 * (2 * (1 * 1)))))
                    # by line (A)
factorial(6) = 720
Python3>>>
```

- Looks just like a hand evaluation

3.3 String Prefixes Example

3.3.1 String Prefixes Definition

- strPrefs() takes a string and returns a string which is the concatenation of all prefixes of the string
- strPrefs("abc") = "abcaba"

```
158 def strPrefs(aStr) :
159   if (aStr == "") :          # line (A)
160     return (aStr + "")
161   else :                     # line (B)
162     return (aStr + strPrefs(aStr[:-1]))
```

- Below is a sample evaluation of strPrefs"abc"
- To evaluate a function applied to actual arguments, we evaluate the body of the function with each formal parameter replaced by the corresponding actual argument
- In the evaluation below we refer to line (A) or 159 and line (B) or 161 (of file Recursion2020PrintCallReturnEval.py)

```
# strPrefs("abc") evaluation
Call 1 -> "abc" + strPrefs("ab")
                    # by line (B)
Call 2 -> "abc" + ("ab" + strPrefs("a"))
                    # by line (B)
Call 3 -> "abc" + ("ab" + ("a" + strPrefs("")))
                    # by line (B)
Call 4 -> "abc" + ("ab" + ("a" + ("" + "")))
                    # by line (A)
strPrefs("abc") = "abcaba"
```

- In writing down an evaluation by hand it is important to hang on to the brackets — they indicate a subexpression that has to be evaluated before an outer expression
- It is possible to write the definition differently to accumulate the prefixes as the computation goes — see later

3.3.2 String Prefixes with Print Statements

```
166 def strPrefsWithPrint(aStr) :
167   return helpStrPrefsWithPrint(aStr,1,"")

169 def helpStrPrefsWithPrint(aStr, call, strAccum) :
170   if (aStr == "") :          # line (A)
171     callMsg = calcStrPrefsCallMsg(call, aStr)
172     print(callMsg)
173     retMsg  = calcStrPrefsRetMsg(call, aStr, strAccum)
174     print(retMsg)
175     return (aStr + strAccum)
176   else :                     # line (B)
177     callMsg = calcStrPrefsCallMsg(call, aStr)
178     print(callMsg)
179     pAccum = helpStrPrefsWithPrint(aStr[:-1],call+1
180                                    ,strAccum)
181     retMsg  = calcStrPrefsRetMsg(call, aStr, pAccum)
182     print(retMsg)
183     return (aStr + pAccum)
```

- Below is the output from `strPrefsWithPrint("abc")` and some comments on the code

```
Python3>>> strPrefsWithPrint("abc")
Call 1 of function with aStr = "abc"
Call 2 of function with aStr = "ab"
Call 3 of function with aStr = "a"
Call 4 of function with aStr = ""
Return call 4 returning "" + ""
Return call 3 returning "a" + ""
Return call 2 returning "ab" + "a"
Return call 1 returning "abc" + "aba"
'abcaba'
Python3>>>
```

- Here `Python3>>>` is my Python shell prompt
- Below are some comments on the code
- `strPrefsWithPrint()`, `helpStrPrefsWithPrint()` description
- `strPrefsWithPrint()` calls `helpStrPrefsWithPrint(n, 1, 1)`
- The arguments to `helpStrPrefsWithPrint()` are:
- `n` initialised to the argument of `strPrefsWithPrint()`
- `call` the call number initialised to `1`
- `strAccum` is the base case (initialised to `""`) — this argument is not needed but there may be generalisations
- The recursive call to `helpStrPrefsWithPrint()` at line `179` increments `call` — mimics a global variable
- The calls to `calcStrPrefsCallMsg()`, `calcStrPrefsRetMsg()` at lines `171`,`177`,`173`,`181` construct the messages which are then printed — we could have lifted the two calls to `calcStrPrefsCallMsg` to just one before the `if`

3.3.3 String Prefixes with Call, Return Messages

- The snag with inserting print statements into code is the program ceases to be a pure function — rather it is a program that takes a string and returns a computation which does some I/O and returns the factorial

- This conflates understanding the original pure function with understanding where the output comes from in code with side effects
- This can be written as a pure function which takes a number and returns a triple of the output string, and lists of call and return messages, and then decide how to present this

```
189 def funcStrPrefsWithMsgs(aStr) :
190   return helpFuncStrPrefsWM(aStr, 1, "", [], [])
```

```
192 def helpFuncStrPrefsWM(aStr, call, strAccum
193                       , callMsgList, retMsgList) :
194   callMsg = calcStrPrefsCallMsg(call, aStr)
195   if (aStr == "") :          # line (A)
196     retMsg  = calcStrPrefsRetMsg(call, aStr, strAccum)
197     return (aStr + strAccum
198           , [callMsg] + callMsgList
199           , [retMsg] + retMsgList )
200   else :                     # line (B)
201     (pAccum, pCallMsgList, pRetMsgList) \
202       = helpFuncStrPrefsWM(aStr[:-1], call+1, strAccum
203                          , callMsgList, retMsgList)
204     retMsg  = calcStrPrefsRetMsg(call, aStr, pAccum)
205     return (aStr + pAccum
206           , [callMsg] + pCallMsgList
207           , [retMsg] + pRetMsgList )
```

- `helpFuncStrPrefsWM()` description — the arguments are:
- `n` initialised to argument of `funcStrPrefsWithMsgs()`
- `call` the call number initialised to `1`
- `strAccum` is the base case (initialised to `""`) — this argument is not needed but there may be generalisations
- `callMsgList`, the call messages list, initialised to an empty list
- `retMsgList`, the return messages list, initialised to an empty list
- The recursive call to `helpFuncStrPrefsWM()` at line `202` increments `call` — mimics a global variable
- The call to `calcStrPrefsCallMsg()` is now before the `if` at line `194` — constructs the call message
- The calls to `calcStrPrefsRetMsg()` at lines `196`,`204` construct the return message

```
209 def calcStrPrefsCallMsg(call, aStr) :
210   return ("Call␣" + str(call)
211           + "␣of␣function␣with␣aStr␣=␣" + pStr(aStr) )

213 def calcStrPrefsRetMsg(call, aStr, acc) :
214   return ("Return␣call␣" + str(call)
215           + "␣returning␣" + pStr(aStr)
216           + "␣+␣" + pStr(acc) )
```

```
72 nl = "\n"

74 sp = "␣"
75 sp20 = sp.join(['']*21) # string of 20 spaces

77 def pStr(aStr) :
78   return ("\"" + aStr + "\"")
```

```
228 def printStrPrefsWM(aStr) :
229   (strPrefsFinal, callMsgList, retMsgList) \
230     = funcStrPrefsWithMsgs(aStr)
```

```
231   print(nl.join(callMsgList)
232     + nl
233     + nl.join(reversed(retMsgList))
234     + nl + "strPrefs("
235     + pStr(aStr) + ")␣=␣"
236     + pStr(strPrefsFinal) )
```

- nl, sp are defined to make the code easier to read
- join() and reversed() are Python built-in functions
- The return messages list is reversed to get the messages in the same order as the print statement version — this might take some thinking
- pStr() is needed since str("") does not produce '""'
- Sample output from printStrPrefsWM("abc")

```
Python3>>> printStrPrefsWM("abc")
Call 1 of function with aStr = "abc"
Call 2 of function with aStr = "ab"
Call 3 of function with aStr = "a"
Call 4 of function with aStr = ""
Return call 4 returning "" + ""
Return call 3 returning "a" + ""
Return call 2 returning "ab" + "a"
Return call 1 returning "abc" + "aba"
strPrefs("abc") = "abcaba"
Python3>>>
```

3.3.4 String Prefixes with Evaluation Messages

- The evaluation messages can also be generated
- The code requires a bit more thought since the evaluation messages are also recursive

```
240 def funcStrPrefsEval(aStr) :
241   return helpFuncStrPrefsEval(aStr, 1, "", [], [])
```

```
243 def helpFuncStrPrefsEval(aStr, call, strAccum
244                        , argList, evalMsgList) :
245   if (aStr == "") :          # line (A)
246     reasonStr = "#␣by␣line␣(A)"
247     evalStr = pStr("")
248     evalMsg  = calcStrPrefsEvalMsg(call, aStr
249                                    , argList, evalStr
250                                    , reasonStr)
251     return (aStr + strAccum
252         , [aStr] + argList, [evalMsg] + evalMsgList )
253   else :                     # line (B)
254     reasonStr = "#␣by␣line␣(B)"
255     evalStr = "strPrefs(" + pStr(aStr[:-1]) + ")"
256     evalMsg  = calcStrPrefsEvalMsg(call, aStr
257                                    , argList, evalStr
258                                    , reasonStr)
259     (pAccum, pArgList, pEvalMsgList) \
260       = helpFuncStrPrefsEval(aStr[:-1], call+1
261                , strAccum, [aStr] + argList, evalMsgList)
262     return (aStr + pAccum
263           , pArgList, [evalMsg] + pEvalMsgList )
```

- helpFuncStrPrefsEval() description — the arguments are:
- n initialised to the argument of funcStrPrefsEval()
- call the call number initialised to 1

- strAccum is the base case (initialised to "") — this argument is not needed but there may be generalisations
- argList, the previous arguments list, initialised to an empty list — the previous arguments are needed to calculate the evaluation message
- evalMsgList, the evaluation messages list, initialised to an empty list
- The recursive call to helpFuncStrPrefsEval() at line 260 increments call — this mimics a global variable
- The calls to calcStrPrefsEvalMsg() at lines 248,256 construct the evaluation message (see description below)

```
265 def calcStrPrefsEvalMsg(call, aStr
266                        , argList, evalStr
267                        , reasonStr) :
268   if (argList == []) :
269     retMsg = ("Call␣" + str(call)
270               + "␣->␣" + pStr(aStr)
271               + "␣+␣" + evalStr
272               + nl + sp20 + reasonStr)
273     return retMsg
274   else :
275     nextStr = argList[0]
276     nextArgList = argList[1:]
277     nextEvalStr \
278       = "(" + pStr(aStr) + "␣+␣" + evalStr + ")"
279     return calcStrPrefsEvalMsg(call, nextStr
280                              , nextArgList, nextEvalStr
281                              , reasonStr)
```

- Notice calcStrPrefsEvalMsg() is recursive unlike calcStrPrefsCallMsg(), calcStrPrefsR
- calcStrPrefsEvalMsg() description — the arguments are:
- call the current call number
- n the current argument
- argList the list of previous arguments
- evalStr the expression on the right hand side of the current operation (see print output below)
- reasonStr is the string identifying the branch of the if used
- If argList is empty, a string is returned
- Otherwise there is a recursive call to wrap the constructed string with the outer arguments (see print output below)
- It is easy to get the arguments in the wrong order — perhaps this is why this function is not given often.

```
293 def printStrPrefsEval(aStr) :
294   (strPrefsFinal, argList, evalMsgList) \
295     = funcStrPrefsEval(aStr)
296   print(nl.join(evalMsgList)
297     + nl + "strPrefs("
298     + pStr(aStr) + ")␣=␣"
299     + pStr(strPrefsFinal) )
```

- Sample output from printStrPrefsEval("abc")

```
Python3>>> printStrPrefsEval("abc")
Call 1 -> "abc" + strPrefs("ab")
                    # by line (B)
```

```
Call 2 -> "abc" + ("ab" + strPrefs("a"))
                    # by line (B)
Call 3 -> "abc" + ("ab" + ("a" + strPrefs("")))
                    # by line (B)
Call 4 -> "abc" + ("ab" + ("a" + ("" + "")))
                    # by line (A)
strPrefs("abc") = "abcaba"
Python3>>>
```

- Looks just like a hand evaluation

3.4 Recursion Examples — Summary

- The evaluation model of substituting actual arguments into the body of a function definition is the easiest for hand evaluation — it helps if there are no side effects or mutated variables
- The messages from print statements may be useful for seeing the pattern of calls or the state of a computation but requires some insight to understand the order of the output
- The examples used are similar to *M269 Python Activities* 3.2 `listSum()`, 3.3 `factorial()`, 3.4 `sumOddNumber()`
- Of course, a real functional programmer would take this a lot further — see The Evolution of a Haskell Programmer
- The point of this piece of Haskell humour is that *Recursion is the GOTO of Functional Programming* and you should really invent the appropriate higher order function to capture your pattern of recursion
- If you want to read further then have a look at An Introduction to Recursion Schemes though be aware that this becomes very quickly more advanced

3.5 Recursion Exercises

- The following exercises ask you to do evaluations of some expression involving a function with a recursive definition
- Do the evaluations in the same style as the above examples
- The Python code is in file `Recursion2020Exercises.py` in the same folder as the notes
- Note that some Python comments contain LaTeX code — this there to generate references
- The python refers to the `timeit` library — see `timeit` — Measure execution time of small code snippets

Activity 1 fib01

- Evaluate `fib01(5)` given the following definition

```
17  def fib01(n) :
18    if n == 0 :
19      return 0       # fib01 (A)
20    elif n == 1 :
```

```
21      return 1       # fib01 (B)
22    else :
23      return (fib01(n - 2) + fib01(n - 1))
24                     # fib01 (C)
```

- Refer to lines (A) or 19, (B) or 21, (C) or 23

Go to Answer

Answer 1 fib01

```
# fib01(5) evaluation
Call 1 -> fib01(3) + fib01(4)          # by line (C)
Calls 2,3 -> (fib01(1) + fib01(2))
             + (fib01(2) + fib01(3))   # by line (C)x2
Calls 4,5,6,7 -> (1 + (fib01(0) + fib01(1)))
                 + ((fib01(0) + fib01(1))
                   + (fib01(1) + fib01(2)))
         # by line (B),(C)x3
Calls 8-13 -> (1 + (0 + 1)))
              + ((0 + 1)
                + (1 + (fib01(0) + fib01(1))))
         # by line (A)x2,(B)x3,(C)
Calls 14-15 -> (1 + (0 + 1)))
               + ((0 + 1)
                 + (1 + (0 + 1)))
         # by line (A),(B)
fib01(5) = 5
```

- This take 15 calls to `fib01()`, 5 calls to `fib01(1)`, 3 calls to `fib01(0)`

Go to Activity

Activity 2 fib02

- Evaluate `fib02(5)` given the following definition

```
28  def fib02(n) :
29    if n == 0 :
30      return 0       # fib02 (A)
31    elif n == 1 :
32      return 1       # fib02 (B)
33    else :
34      return (fib02List(n,0,1)[n])
35                     # fib02 (C)

37  def fib02List(n,first,second) :
38    if n == 0 :
39      return [first] # fib02List (D)
40    else :
41      return ([first]
42              + fib02List(n - 1, second, first + second))
43                     # fib02List (E)
```

Go to Answer

Answer 2 fib02

```
# fib02(5) evaluation
Call 1 -> fib02List(5,0,1)[5]  # by line fib02 (C)
fib02List(5,0,1) -> ([0]       # by line fib02List (E)
                     + fib02List(4,1,1))
fib02List(4,1,1) -> ([1]       # by line fib02List (E)
                     + fib02List(3,1,2))
fib02List(3,1,2) -> ([1]       # by line fib02List (E)
                     + fib02List(2,2,3))
fib02List(2,2,3) -> ([2]       # by line fib02List (E)
                     + fib02List(1,3,5))
fib02List(1,3,5) -> ([3]       # by line fib02List (E)
                     + fib02List(0,5,8))
fib02List(0,5,8) -> [5]        # by line fib02List (D)
fib02List(5,0,1) = [0, 1, 1, 2, 3, 5]
```

```
fib02(5) = 5
```

- This takes 6 calls to `fib02List`

Go to Activity

- From How to Compute Fibonacci Numbers? and Fibonacci Formulae

```
fib (n + k) = fib (n-1) * fib k + fib n * fib (k+1)
```

- Proof by induction

```
fib (n + 2 + k)
  = fib (n+k) + fib (n+k+1)                    -- by fib
  = fib (n-1) * fib k + fib n * fib (k+1)
    + fib n * fib k + fib (n+1) * fib (k+1)    -- by hyp
  = fib (n+1) * fib k + fib (n+2) * fib (k+1) -- by fib
```

- We now derive a method to compute `fib` in O(log n)

```
fib (2n+1) = (fib n)^2 + (fib (n+1))^2    -- (1)
fib (2n+2) = fib n * fib (n+1) + fib (n+1) * fib (n+2)
  = fib n * fib (n+1) + fib (n+1) * (fib n + fib (n+1))
  = 2 * fib n * fib (n+1) + (fib (n+1))^2  -- (2)
fib 2n = 2 * fib n * fib (n+1) - (fib n)^2 -- (3)
  -- (3) = (2) - (1)
```

```
fib2v :: Integer -> (Integer,Integer)
fib2v 0 = (0,1)
fib2v n
  | n `mod` 2 == 0  = (c,d)
  | otherwise       = (d,c+d)
  where
  (a,b) = fib2v (n `div` 2)
  c     = 2 * a * b - a * a
  d     = a * a + b * b
```

Activity 3 fibv

- Evaluate `fibv(5)` given the following definition

```
def fibv(n) :
  if n == 0 :
    return 0       # fibv (A)
  else :
    return (fib2v(n - 1)[1])
                   # fibv (B)

def fib2v(n) :
  if n == 0 :
    return (0,1)   # fib2v (C)
  else :
    (a,b) = fib2v(n // 2)
    c = 2 * a * b - a * a
    d = a * a + b * b
    if (n % 2 == 0) :
      return (c,d) # fib2v (D)
    else :
      return (d, c + d)
                   # fib2v (E)
```

Go to Answer

Answer 3 fibv

```
# fibv(5) evaluation
fibv(5) -> fib2v(4)[1]  # by fibv (B)
fib2v(4) -> (c,d)       # by fib2v (D)
  where (a,b) = fib2v(2)
        c = 2 * a * b - a * a
```

```
        d = a * a + b * b
fib2v(2) -> (c,d)       # by fib2v (D)
  where (a,b) = fib2v(1)
        c = 2 * a * b - a * a
        d = a * a + b * b
fib2v(1) -> (d, c + d)  # by fib2v (E)
  where (a,b) = fib2v(0)
        c = 2 * a * b - a * a
        d = a * a + b * b
fib2v(0) -> (0,1)       # by fib2v (C)
-> fib2v(1) = (1,1)     # by arithmetic
-> fib2v(2) = (1,2)
-> fib2v(4) = (3,5)
-> fibv(5) = 5
```

Go to Activity

Example Timings

```
Python3>>> timeit(lambda: fib01(40),number=1)
41.122885193966795
Python3>>> timeit(lambda: fib02(40),number=1)
3.125402145087719e-05
Python3>>> timeit('fib02(40)',globals=globals())
16.959519409982022
Python3>>> timeit('fib02(40)',globals=globals(),number=1)
2.4509034119546413e-05
Python3>>> timeit(lambda: fibv(40),number=1)
6.077397847548127e-05
Python3>>> timeit(lambda: fibv(200),number=1)
2.392695751041174e-05
Python3>>> timeit(lambda: fib02(200),number=1)
0.00042063603177666664
Python3>>> timeit(lambda: fibv(40),number=1)
7.555994670838118e-06
Python3>>> timeit(lambda: fibv(40),number=1)
7.733004167675972e-06
Python3>>> fib02(200)
280571172992510140037611932413038677189525
Python3>>> fibv(200)
280571172992510140037611932413038677189525
```

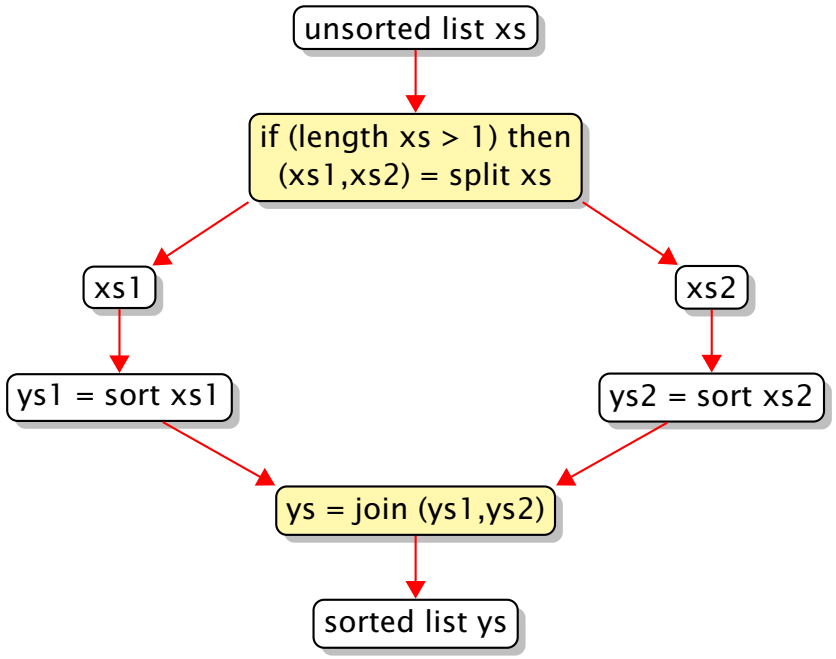
```
GHCi> :set +s
GHCi> fib02 40
102334155
(0.00 secs, 89,456 bytes)
GHCi> fib02 200
280571172992510140037611932413038677189525
(0.00 secs, 186,992 bytes)
GHCi> fibv 200
280571172992510140037611932413038677189525
(0.00 secs, 104,528 bytes)
GHCi> fib01 40
^CInterrupted.
GHCi> fibv 300
222232244629420445529739893461909967206666939096499764990979600
(0.01 secs, 121,544 bytes)
```

```
GHCi> fib01 30
832040
(1.74 secs, 921,589,368 bytes)
GHCi> fib01 33
3524578
(7.31 secs, 3,903,684,264 bytes)
GHCi> fib01 34
5702887
(12.44 secs, 6,316,252,408 bytes)
GHCi> fib01 35
9227465
(19.20 secs, 10,219,870,536 bytes)
GHCi> fib01 36
14930352
```

```
(31.08 secs, 16,536,056,616 bytes)
GHCi> fib01 37
24157817
(49.99 secs, 26,755,861,504 bytes)
GHCi> fib01 38
39088169
(84.35 secs, 43,291,848,464 bytes)
```

4 Sorting

4.1 Taxonomy of Comparison Sorts



Other Sorting Algorithm Classifications

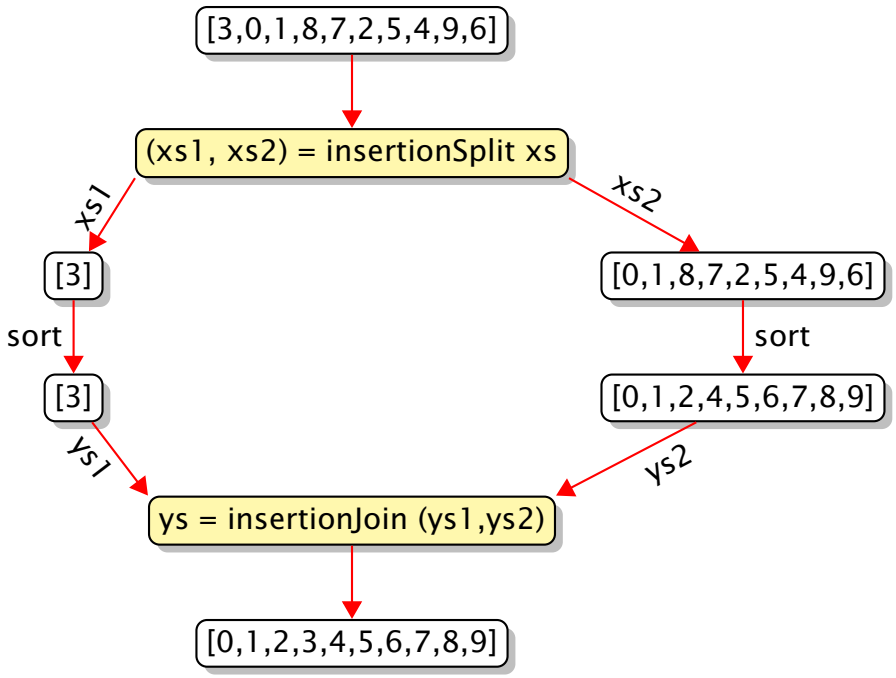
- See Wikipedia Sorting algorithm for big list
- *Comparison Sorts*
  - Insertion sort, Selection sort, Merge sort, Quicksort, Bubble sort
  - Sorting via a data structure: Tree sort, Heap sort
- *Non-Comparison sorts* — distribution sorts — bucket sort, radix sort
- *Sorts used in Programming Language Libraries*
  - Timsort by Tim Peters — used in Python and Java — combination of merge and insertion sorts
  - Haskell — modified Mergesort by Ian Lynagh in GHC implementation

4.2 Insertion Sort

4.2.1 Insertion Sort — Abstract Algorithm

- *Insertion Split* xs1 is the singleton list of the first item; xs2 is the rest of the list
- *Insertion Join* insert the item in the singleton list into the sorted result of the rest of the list

Abstract Sorting Algorithm Diagram for Insertion Sort



4.2.2 Insertion Sort — Python

```
4  def insSort(xs):
5    if len(xs) <= 1:
6      return xs
7    else:
8      return ins(xs[0],insSort(xs[1:]))

10 def ins(x,xs):
11   if xs == []:
12     return [x]
13   elif x <= xs[0]:
14     return [x] + xs
15   else:
16     return [xs[0]] + ins(x,xs[1:])
```

Python Rewritten

- In the style of the abstract algorithm

```
20 def insSort01(xs):
21   if len(xs) <= 1:
22     return xs
23   else:
24     (xs1,xs2) = insertionSplit(xs)
25     ys1 = insSort01(xs1)
```

```
26      ys2 = insSort01(xs2)
27      ys = insertionJoin(ys1,ys2)
28      return ys

30  def insertionSplit(xs):
31    (xs1,xs2) = (xs[0:1],xs[1:])
32    return  (xs1,xs2)

34  def insertionJoin(ys1,ys2):
35    if ys2 == []:
36      return ys1
37    elif ys1[0] <= ys2[0]:
38      return ys1 + ys2
39    else:
40      return ys2[0:1] + insertionJoin(ys1,ys2[1:])
```

4.2.3 Insertion Sort — Haskell

```
1   module M269Prsntn2018JTutorialUnits345 where
2     import Data.List
3     import Data.Maybe
```

1. A Haskell script starts with a module header which starts with the reserved identifier, `module` followed by the module name, `M269TutorialSorting`
2. The module name must start with an upper case letter and is the same as the file name (without its extension of *.lhs*)
3. Haskell uses *layout* (or the *off-side rule*) to determine scope of definitions, similar to Python
4. The body of the module follows the reserved identifier `where` and starts with two `import` declarations
5. These import the built-in libraries `Data.List` and `Data.Maybe`
6. We use the `sort` function from `Data.List`.
7. The `Maybe` datatype from `Data.Maybe` will be used at the end of this script to implement the trees with data.

```
5     insSort  []  = []
6     insSort  [x] = [x]
7     insSort (x : xs)  =  ins x (insSort xs)

9     ins  x  []  =  [x]
10    ins  x  (y:ys)
11      =  if x <= y
12         then x:y:ys
13         else y : (ins x ys)
```

- For *structured English*, I have used a subset of *Haskell* (`http://haskell.org`) — in the code above:
- *insSort* and *ins* are function defined by several equations
- We use *indentation* to determine scope — see Landin (1966) and Python (see `http://docs.python.org/2/tutorial/introduction.html#first-steps-towards-programmi` in the *Python Tutorial*)
- Function application is denoted by juxtaposition and is more tightly binding than (almost) anything else

- we write `f x` and not `f (x)`
- `f x y` means `(f x) y`

This notational convention has huge advantages — discuss and also see http://en.wikipedia.org/wiki/Curried_function and http://slid.es/gsklee/functional-programming-in-5-minutes (which does it in JavaScript, worth a look)

- Lists are denoted with brackets `[1,2,3]`, the empty list is `[]`
- `(:)` is the operator that prefixes an element to a list, `1:[2,3] == [1,2,3]`
- Parentheses over-ride precedence

4.2.4 Activity 2 — Insertion Sort: Trace an Evaluation

Insertion Sort — Haskell Recursive

- Evaluation of `insSort [3,0,1,8,7]`

	Expression to Evaluate	Reason
	`insSort [3,0,1,8,7]`	Initial
→	`ins 3 (insSort [0,1,8,7])`	line 7
→	`ins 3 (ins 0 (insSort [1,8,7]))`	line 7
→	`ins 3 (ins 0 (ins 1 (insSort [8,7])))`	line 7
→	`ins 3 (ins 0 (ins 1 (ins 8 (insSort [7]))))`	line 7
→	`ins 3 (ins 0 (ins 1 (ins 8 [7])))`	line 6
→	`ins 3 (ins 0 (ins 1 (7:(ins 8 []))))`	line 13
→	`ins 3 (ins 0 (ins 1 (7:[8])))`	line 9
→	`ins 3 (ins 0 (ins 1 [7,8]))`	(:) operator
→	`ins 3 (ins 0 (1:7:[8]))`	line 12
→	`ins 3 (ins 0 [1,7,8])`	(:) operator
→	`ins 3 (0:1:[7,8])`	line 12
→	`ins 3 [0,1,7,8]`	(:) operator
→	`0:(ins 3 [1,7,8])`	line 13
→	`0:(1:(ins 3 [7,8]))`	line 13
→	`0:(1:(3:7:[8]))`	line 12
→	`[0,1,3,7,8]`	(:) operator

- Note that the evaluation consumes more space in the process of evaluation;
- also note that you need to be careful with the brackets when doing an evaluation like this by hand.

Insertion Sort — Python Recursive

- Evaluation of `insSort([3,0,1,8,7])`

	Expression to Evaluate	Reason
	insSort([3,0,1,8,7])	Initial
→	ins(3, insSort([0,1,8,7]))	line 7
→	ins(3, ins(0, insSort([1,8,7])))	line 7
→	ins(3, ins(0, ins(1, insSort([8,7]))))	line 7
→	ins(3, ins(0, ins(1, ins(8, insSort([7])))))	line 7
→	ins(3, ins(0, ins(1, ins(8, [7]))))	line 5
→	ins(3, ins(0, ins(1, ([7] + ins(8, [])))))	line 15
→	ins(3, ins(0, ins(1, ([7] + [8]))))	line 11
→	ins(3, ins(0, ins(1, [7,8])))	(+) operator
→	ins(3, ins(0, ([1] + [7,8])))	line 13
→	ins(3, ins(0, [1,7,8]))	(+) operator
→	ins(3, ([0] + [1,7,8]))	line 13
→	ins(3, [0,1,7,8])	(+) operator
→	[0] + (ins 3 [1,7,8])	line 15
→	[0] + ([1] + (ins 3 [7,8]))	line 15
→	[0] + ([1] + ([3] + ([7,8])))	line 13
→	[0,1,3,7,8]	(+) operator

- Note that the evaluation consumes more space in the process of evaluation;
- also note that you need to be careful with the brackets when doing an evaluation like this by hand.

4.2.5 Insertion Sort — Non-recursive

- The non-recursive version of *Insertion* sort takes each element in turn and inserts it in the ordered list of elements before it.

```
for index = 1 to (len(xs)-1) do
  insert xs[index] in order in xs[0..index-1]
```

- Here is a Python implementation of the above (based on Miller and Ranum (2011, page 215)).
- It uses the *Python Style Guide PEP 8* http://www.python.org/dev/peps/pep-0008/ (Python Enhancement Proposals) — I must admit I prefer indenting with 2 spaces but I imagine the M269 module will have its own guidelines.

```
44 def insertionSort(xs):
45   for index in range(1, len(xs)):
46     currentValue = xs[index]
47     position = index
48     while (position > 0) and xs[position - 1] > currentValue:
49       xs[position] = xs[position - 1]
50       position = position - 1

52     xs[position] = currentValue
```

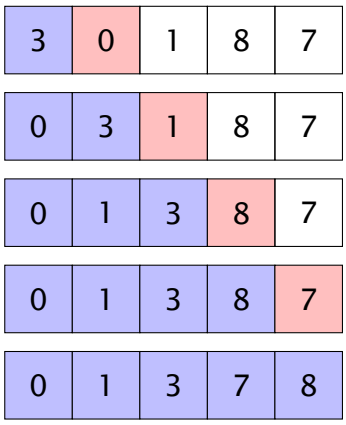
4.2.6 Activity 3 — Insertion Sort Non-recursive Trace

Insertion Sort — Python Non-recursive

- Evaluation of `insertionSort([3,0,1,8,7])`

- Showing just the outer `for index` loop
- 

start array



5 Searching

5.1 Binary Search

- Given an ordered list (xs) and a value (`val`), return
  - Position of `val` in `xs` or
  - Some indication if `val` is not present
- *Simple strategy:* check each value in the list in turn
- *Better strategy:* use the ordered property of the list to reduce the range of the list to be searched each turn
  - Set a range of the list
  - If `val` equals the mid point of the list, return the mid point
  - Otherwise half the range to search
  - If the range becomes negative, report not present (return some distinguished value)
- *Binary Search* in various forms

Binary Search Iterative

```
5  def binarySearchIter(xs,val):
6    lo = 0
7    hi = len(xs) - 1

9    while lo <= hi:
10     mid = (lo + hi) // 2
11     guess = xs[mid]

13     if val == guess:
14       return mid
15     elif val < guess:
16       hi = mid - 1
17     else:
```

```
18        lo = mid + 1

20    return None
```

Binary Search Recursive

```
24  def binarySearchRec(xs,val,lo=0,hi=-1):
25    if (hi == -1):
26      hi = len(xs) - 1

28    mid = (lo + hi) // 2

30    if hi < lo:
31      return None
32    else:
33      guess = xs[mid]
34      if val == guess:
35        return mid
36      elif val < guess:
37        return binarySearchRec(xs,val,lo,mid-1)
38      else:
39        return binarySearchRec(xs,val,mid+1,hi)
```

5.1.1 Binary Search Exercise

Given the *Python* definition of `binarySearchRec` from above, trace an evaluation of `binarySearchRec(xs, 25)` where `xs` is

```
xs = [2,5,7,15,17,19,21,24,27,31,37,48,57,87,95]
```

Binary Search — Solution

```
xs = [2,5,7,15,17,19,21,24,27,31,37,48,57,87,95]
binarySearchRec(xs, 25)
xs = [2,5,7,15,17,19,21,24,27,31,37,48,57,87,95]
binarySearchRec(xs,25,8,14) by line 38
xs = [2,5,7,15,17,19,21,24,27,31,37,48,57,87,95]
binarySearchRec(xs,25,8,10) by line 38
xs = [2,5,7,15,17,19,21,24,27,31,37,48,57,87,95]
binarySearchRec(xs,25,8,8) by line 36
xs = [2,5,7,15,17,19,21,24,27,31,37,48,57,87,95]
binarySearchRec(xs,25,8,7) by line 36
Return value: None by line 30
```

Binary Search — Comparison

- Both forms compare the given value (`val`) to the mid-point value of the range of the list (`xs[mid]`)
- If not found, the range is adjusted via assignment in a `while` loop (iterative) or function call (recursive)
- The recursive version has *default parameter* values to initialise the function call (evil, should be a helper function)
- There are two base cases:

  - The value is found (`val == guess`)
  - The range becomes negative (`hi < lo`)
- The return value is either `mid` or None
- What is the *type* of the binary search function ?
- How do we *reason* about the code ?

5.1.2 Binary Search Error

- Although the basic idea of binary search is comparatively straightforward, the details can be surprisingly tricky

  (Knuth, 1998, section 6.2.1)
- The real problem is that programmers have spent far too much time worrying about efficiency in the wrong places and at the wrong times; premature optimization is the root of all evil (or at least most of it) in programming.

  (Knuth, 1974, 671)
- See other Donald Knuth quotes at Wikiquote: Donald Knuth

```
46 def binarySearchError01(xs,target):
47   lo = 0
48   hi = len(xs)

50   while True :
51     centre = (hi + lo)//2
52     if centre == lo :
53       # Only two items left to test so it is either
54       # centre or centre + 1 or it is not in. This is
55       # an exit point of the infinite loop
56       return (centre if xs[centre] == target
57               else centre+1 if xs[centre+1] == target
58               else -1)
59     if xs[centre] < target :
60       lo = centre
61     elif target < xs[centre] :
62       hi = centre
63     else :
64       return centre
```

- Suppose `xs = [10,20]`, `target = 30`
- Trace the execution of the above code and identify a run time error
- This is exercise 4.6 in Backhouse (2003, pages 50–51,269–270) which refers to the text Winder and Roberts (1998, page 592), which was repeated in Winder and Roberts (2000, page 561)

	lo	hi	centre	xs[centre]
Before loop	0	2	-	-
Loop 1	0	2	1	20
centre == lo	False			
xs[centre] < target	True			
Loop 2	1	2	1	20
centre == lo	True			
	Crash			

- `xs = [10,20]`, `target = 30`
- And we get an out of range error
- How should it be corrected ?
- Throw it away and start again

5.1.3 Binary Search — Position

- M269 2017J TMA02 Q1 provides an iterative function to return the position where a target is or should be inserted.
- Insert 25 into [10,20,30] should return [10,20,25,30]
- Insert 20 into [10,20,30] should return [10,20,20,30]
- Insert 5 into [10,20,30] should return [5,10,20,30]
- Insert 35 into [10,20,30] should return [10,20,30,35]
- TMA02 Q1(b) specifies the function, `binSchPosnIterTMA` to return the position
- **Inputs** *xs* sequence of n integers $x_0, x_1, \ldots, x_{n-1}$
- (*first*,*last*) index of first and last items in sorted subsequence
- *target* integer to find insertion position
- **Pre** $0 \leqslant \text{first} < n$ and $\text{last} < n$
- If *first* $\leqslant$ *last* then $x_{\text{first}} \leqslant x_{\text{first+1}} \leqslant \cdots \leqslant a_{\text{last}}$
- **Post** If *first* > *last* then return value, *pos*, is *first*
- If *first* $\leqslant$ *last* and *target* > x_{last} then *pos* = *last* + 1
- If *first* $\leqslant$ *last* and *target* $\leqslant x_{\text{last}}$ then *pos* is the smallest value such that *first* $\leqslant$ *pos* $\leqslant$ *last* and *target* $\leqslant x_{\text{pos}}$

```
68 def binSchPosnIterTMA(xs,first,last,target) :
69   while first <= last :
70     middle = (first + last) // 2
71     if xs[middle] == target :
72       return middle
73     if target < xs[middle] :
74       last = middle - 1
75     else :
76       first = middle + 1
77   return first
```

- **Evaluate**
- `binSchPosnIterTMA([10,20,30],0,2,25)`
- `binSchPosnIterTMA([10,20,30],0,2,20)`
- `binSchPosnIterTMA([10,20,30],0,2,5)`
- `binSchPosnIterTMA([10,20,30],0,2,35)`

```
Python3>>> binSchPosnIterTMA([10,20,30],0,2,25)
2
Python3>>> binSchPosnIterTMA([10,20,30],0,2,20)
1
Python3>>> binSchPosnIterTMA([10,20,30],0,2,5)
```

```
0
Python3>>> binSchPosnIterTMA([10,20,30],0,2,35)
3
Python3>>>
```

- Are these tests enough ?
- What if we have sequences of items with the same key as the target ?
- **Evaluate**
- `binSchPosnIterTMA([10,20,30],0,2,20)`
- `binSchPosnIterTMA([10,20,20,30],0,3,20)`
- `binSchPosnIterTMA([10,20,20,20,30],0,4,20)`
- `binSchPosnIterTMA([10,20,20,20,20,30],0,5,20)`

```
Python3>>> xs2
[10, 20, 20, 30]
Python3>>> binSchPosnIterTMA(xs2,0,3,20)
1
Python3>>> xs3
[10, 20, 20, 20, 30]
Python3>>> binSchPosnIterTMA(xs3,0,4,20)
2
Python3>>> xs4
[10, 20, 20, 20, 20, 30]
Python3>>> binSchPosnIterTMA(xs4,0,5,20)
2
Python3>>> xs5
[10, 20, 20, 20, 20, 20, 30]
Python3>>> binSchPosnIterTMA(xs5,0,6,20)
3
Python3>>>
```

- Does this meet the specification ?
- Why not ?

Binary Search and Out by One Errors

- *Out by One* errors are common in implementation of Binary Search (or similar errors)
- The specification is best thought about as partitioning the original list so the insertion point is below or to the left of any identical entries
- So we write the postcondition to say that:
- $x_0, x_1, \ldots x_{pos-1} <$ target
- $x_{pos}, \ldots, x_{n-1} \geqslant$ target
- where n is the length of the subsequence and *pos* is the insertion point returned by the function
- Getting any of the above numbers *out by one* or something similar, leads to bugs

Loop invariant

- To find *pos* we are going to maintain two parts of the subsequence
- $x_0, x_1, \ldots x_{first-1} <$ target

- $x_{last+1}, \ldots, x_{n-1} \geqslant$ target
- We start with *first* as 0 and *last* as (n - 1)
- So at the start the two partitions are empty (and hence the conditions above are True)
- The `while` loop maintains the loop invariant above, while moving *first* and *last*
- At the end of the loop, last = first – 1
- Hence (with some arithmetic) we can check the post condition
- Below is `binSchPosnIterBelow` which implements this

Insertion Point Below Sequence of Identical Items

```
81 def binSchPosnIterBelow(xs,first,last,target) :

83   while first <= last :
84     # Invariant:
85     # Items in xs[:first] < target
86     # Items in xs[last+1:] >= target
87     middle = (first + last) // 2
88     if xs[middle] < target :
89       first = middle + 1
90     else :
91       last = middle - 1

93   return first
```

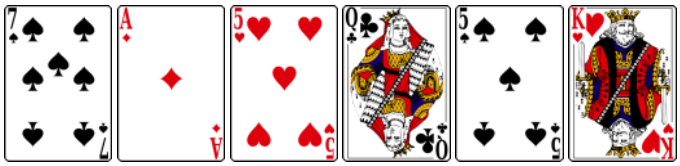
Testing binSchPosnIterBelow()

```
Python3>>> xs2
[10, 20, 20, 30]
Python3>>> binSchPosnIterBelow(xs2,0,3,20)
1
Python3>>> xs3
[10, 20, 20, 20, 30]
Python3>>> binSchPosnIterBelow(xs3,0,4,20)
1
Python3>>> xs4
[10, 20, 20, 20, 20, 30]
Python3>>> binSchPosnIterBelow(xs4,0,5,20)
1
Python3>>> xs5
[10, 20, 20, 20, 20, 20, 30]
Python3>>> binSchPosnIterBelow(xs5,0,6,20)
1
Python3>>>
```

- Does this meet the specification ?

Insertion Sort is a *stable* sort

- Insertion sort is a *stable* sort
- Items with identical keys are sorted in the same order as they appear in the input



- If the aove cards are sorted by value as the key (ignoring suit) then the 5 of Spades should stay above (or to the right) of the 5 of Hearts
- `binSchPosnIterBelow()` would not do that
- How should we modify the algorithm to produce `binSchPosnIterAbove()` ?

Insertion Point Above Sequence of Identical Items

```
97  def binSchPosnIterAbove(xs,first,last,target) :

99    while first <= last :
100     # Invariant:
101     # Items in xs[:first] <= target
102     # Items in xs[last+1:] > target
103     middle = (first + last) // 2
104     if xs[middle] <= target :
105       first = middle + 1
106     else :
107       last = middle - 1

109   return first
```

Testing binSchPosnIterAbove()

```
Python3>>> xs2
[10, 20, 20, 30]
Python3>>> binSchPosnIterAbove(xs2,0,3,20)
3
Python3>>> xs3
[10, 20, 20, 20, 30]
Python3>>> binSchPosnIterAbove(xs3,0,4,20)
4
Python3>>> xs4
[10, 20, 20, 20, 20, 30]
Python3>>> binSchPosnIterAbove(xs4,0,5,20)
5
Python3>>> xs5
[10, 20, 20, 20, 20, 20, 30]
Python3>>> binSchPosnIterAbove(xs5,0,6,20)
6
Python3>>>
```

- Does this meet the *new* specification ?

Insertion Point Below Sequence of Identical Items — Recursive

- As before the function must be called with a sorted subsequence of *xs* with *first* and *last* being the indices of the sorted part start and end
- Depending on the relation of `xs[middle]` to `target` we recurse on either the beginning or ending half of the subsequence
- The base case is where `first` and `last` cross over
- Notice how the recursive version follows the loop invariant

Insertion Point Below Sequence of Identical Items — Recursive — Implementation

```
113 def binSchPosnRecBelow(xs,first,last,target) :
115   middle = (first + last) // 2
117   if first > last :
118     return first
119   elif xs[middle] < target :
120     return binSchPosnRecBelow(xs,middle+1,last,target)
121   else :
122     return binSchPosnRecBelow(xs,first,middle-1,target)
```

Testing binSchPosnRecBelow()

```
Python3>>> xs2
[10, 20, 20, 30]
Python3>>> binSchPosnRecBelow(xs2,0,3,20)
1
Python3>>> xs3
[10, 20, 20, 20, 30]
Python3>>> binSchPosnRecBelow(xs3,0,4,20)
1
Python3>>> xs4
[10, 20, 20, 20, 20, 30]
Python3>>> binSchPosnRecBelow(xs4,0,5,20)
1
Python3>>> xs5
[10, 20, 20, 20, 20, 20, 30]
Python3>>> binSchPosnRecBelow(xs5,0,6,20)
1
Python3>>>
```

- Does this meet the specification ?

Insertion Point Above Sequence of Identical Items — Recursive

- We have the same small change to the function to produce the Above version
- Note that small changes to the function can make a big difference

```
126 def binSchPosnRecAbove(xs,first,last,target) :
128   middle = (first + last) // 2
130   if first > last :
131     return first
132   elif xs[middle] <= target :
133     return binSchPosnRecAbove(xs,middle+1,last,target)
134   else :
135     return binSchPosnRecAbove(xs,first,middle-1,target)
```

- Note that the TMA02 Q1 version is slightly different to both the Below and Above versions

Testing binSchPosnRecAbove()

```
Python3>>> xs2
[10, 20, 20, 30]
Python3>>> binSchPosnRecAbove(xs2,0,3,20)
3
Python3>>> xs3
[10, 20, 20, 20, 30]
Python3>>> binSchPosnRecAbove(xs3,0,4,20)
4
Python3>>> xs4
[10, 20, 20, 20, 20, 30]
```

```
Python3>>> binSchPosnRecAbove(xs4,0,5,20)
5
Python3>>> xs5
[10, 20, 20, 20, 20, 20, 30]
Python3>>> binSchPosnRecAbove(xs5,0,6,20)
6
Python3>>>
```

- Does this meet the *new* specification ?

6 String Search

- *Sunday Quick Search* algorithm
- *Knuth-Morris-Pratt (KMP)* algorithm
- Exact String Matching Algorithms — animations in Java

6.1 Sunday Quick Search Algorithm

Example

- Sunday Quick Search example from M269 2014J TMA02
- Consider the alphabet C, G, A, T and a target string CGTACTCGTAGT.
- Calculate the shift table for this search, explaining in detail how you used the part of the algorithm that builds the table to derive your results.
- Given the target string CGTACTCGTAGT, the search string CGTACTCGGCGTAAAGTGCGTCTT and the shift table calculated above
- describe, with diagrams if necessary, how the first attempt to match the target string against the search string fails
- and how the target string slides along the search string for the second matching attempt.

Sunday Quick Search Shift Table

- The shift table for the Sunday Quick Search algorithm:
  - If the character does not appear in the target string *T*, the shift distance is one more than the length of *T*
  - If the character does appear in *T* the shift distance is the first position at which it appears, counting from right to left and starting at 1
- Shift table:

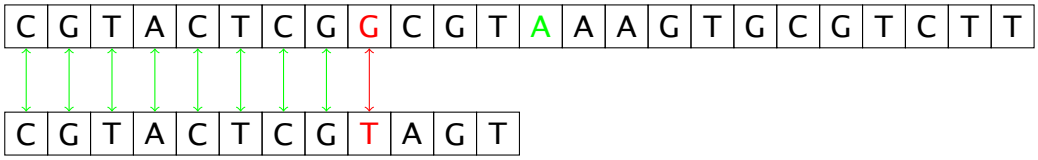
A	C	G	T
3	6	2	1

- See the example at Quick Search algorithm

Calculating the Shift

- Given the following alignment of *Search* and *Target* strings

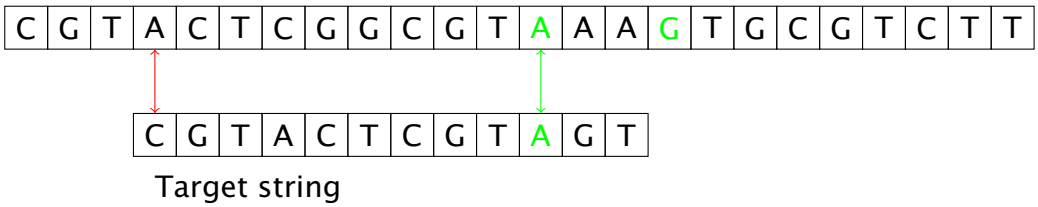
Search string



Target string

- The next character in the *Search* string after the *Target* string is A so the shift will be 3 places

Search string



Target string

6.2 Knuth-Morris-Pratt (KMP) Algorithm

Example

- Knuth-Morris-Pratt (KMP) algorithm example from M269 2014J TMA02
- Consider the alphabet C, G, A, T and a target string CGTACTCGTAGT.
- Calculate the prefix table for this target string,
- explaining in detail how you derived the sixth, seventh and eighth entries in your table.

KMP Prefix Table

- The KMP *prefix table* takes the *target string*, t, and a *position*, p, in t and returns an integer k
- k is the length of the maximum proper prefix of t up to position p that is a suffix of t up position p
- We define a prefix function to take a *target string*, t, and a number $k \geqslant 0$, which is the number of items off the front of t
- Formally, if our position index is 0 based we have:
- prefixTable(t, 0) = 0
- prefixTable(t, p)

  $= \max\{k : k \leqslant p \wedge [\text{prefix}(t, k) \sqsupset \text{prefix}(t, p + 1)]\}$
- Here $\sqsupset$ means *proper suffix* — that is a suffix that does not include the whole string (Cormen et al., 2009, page 986)

KMP Shift Function

- M269 does not give the KMP shift function (or table) — we give it here for interest
- The shift function takes the *target string,* t, and the number of characters in the target string that have been matched, q and returns the shift.
- shift(t, 0) = 1
- shift(t, q) = q – prefixTable(t, q – 1)
- This assumes 0 based list indexing
- The notation here comes from Cormen et al. (2009, 2009, section 32.4, page 1004) — this uses 1 based list indexing and also provides code.
- Cormen uses the terminology *Text* for *Search string* and *Pattern* for *Target string*

KMP Prefix and Shift Table

	C	G	T	A	C	T	C	G	T	A	G	T	Target string
	0	1	2	3	4	5	6	7	8	9	10	11	Position (Index), p
0	1	2	3	4	5	6	7	8	9	10	11	12	Match, q
	0	0	0	0	1	0	1	2	3	4	0	0	prefixTable(t, p)
1	1	2	3	4	4	6	6	6	6	6	11	12	shift(t, q)

7 Bags

7.1 Bags: Definitions

- **Bag** unordered collection that may contain duplicate items
- Also known as Multiset
- **Multiplicity** of an element is the number of instances of an element
- A Bag or Multiset may be defined as a two-tuple (A, m) where A is the underlying set from which elements are drawn, and a function $m : A \to \mathbb{N}$
- Note that some definitions exclude 0 from the range of m so it would be denoted $m : A \to \mathbb{N}_{\geqslant 1}$

Bag: Example

- 120 has prime factorization $120 = 2^3 \times 3^1 \times 5^1$
- Gives bag $\{2, 2, 2, 3, 5\}$

- The {} is an abuse of the usual set notation
- Representations:
- Sorted list [2, 2, 2, 3, 5]
- Unsorted list [2, 3, 2, 5, 2]
- Sorted list of pairs [(2, 3), (3, 1), (5, 1)]

house

Bag: Operations

- **Create** an empty bag
- **Queries**
  - `isEmptyBag`
  - `sizeBag` total number of elements
  - `elemOccurs` number of an element
  - `elemMember` is there at least one copy of an element
- **Construction**
  - `insertElem` insert one copy of an element
  - `insertMany` insert a number of copies
  - `deleteElem` delete a single copy of an element
  - `deleteMany` delete a number of copies
  - `deleteAll` deleteAll all copies

M269 2018J TMA02 Bag Operations

- `Bag()` creates a new empty bag
- `add(self, elem)` adds one copy of `elem` to the bag `self`
- `count(self,elem)` number of `elem` in the bag `self`
- `size(self)` total number of copies in `self`
- `clear(self,elem)` removes all copies of `elem`
- `ordered(self)` returns contents of bag `self` as a list of pairs `(count, elem)` in decreasing order of `count`

7.2 Bags: Implementations

- Python Counter is a subclass of dict which is Python's version of bags or multisets
- Data.MultiSet is Haskell's implementation of multisets or bags — it is based on Data.Map

- Multiset describes the ADT multiset or bag in several programming languages

Python `dict`

- A mapping object or dictionary maps hashable values to arbitrary objects
- A dictionary is a mutable object
- **Creation**

```
aD = dict()      # dictionary constructor
aD = {}          # literal empty dictionary
aD = {'to': 2, 'be': 2, 'or': 1, 'not': 1}
                 # literal key:value pairs
```

- Creation methods

```
Python3>>> aD = {'to': 2, 'be': 2
...              , 'or': 1, 'not': 1}
Python3>>> bD = dict(to=2, be=2, orA=1, notA=1)
Python3>>> cD = dict(zip(['to','be','or','not']
...                      ,[2,2,1,1]))
Python3>>> dD = dict([('to',2),('be',2)
...                  ,('or',1),('not',1)])
Python3>>> eD = dict({'to': 2, 'be': 2
...                  , 'or': 1, 'not': 1})
Python3>>> aD == cD == dD == eD
True
```

- Keywords in keyword arguments must not clash with built-in keywords
- Implicit line joining means we can split expressions in parentheses, square brackets or curly braces over more than one physical line without using backslashes.

Queries

- Queries in the *M269 Companion*

```
Python3>>> aD = {'to': 2, 'be': 2
...              , 'or': 1, 'not': 1}
Python3>>> len(aD)
4
Python3>>> key = 'be'
Python3>>> key in aD
True
Python3>>> aD[key]
2
Python3>>> aD.keys()
dict_keys(['to', 'or', 'not', 'be'])
Python3>>> list(aD.keys())
['to', 'or', 'not', 'be']
Python3>>> aD.items()
dict_items([('to', 2), ('or', 1)
           , ('not', 1), ('be', 2)])
Python3>>> list(aD.items())
[('to', 2), ('or', 1), ('not', 1), ('be', 2)]
Python3>>> ('to',2) in aD.items()
True
```

Total Exercise

Activity 4 Total Exercise

- Write a function, `totalValue`, that takes a dictionary, `xD`, and returns the total of all the values of the key:value pairs (Assumes value is a numeric type)

Go to Answer

Answer 4 Total Exercise

- Answer 4 Total Exercise

```
def totalValue(xD) :
  """
  totalValue takes a dictionary, xD,
  and returns the total of all the values
  of the key:value pairs
  Assumes value is a numeric type
  """

  tVal = sum([v for (k,v) in xD.items()])
  return tVal

  # Alternative
  # tVal = sum(xD.values())
```

- Answer 4 Total Exercise — sample use

```
Python3>>> aD
{'not': 1, 'be': 2, 'to': 2, 'or': 1}
Python3>>> totalValue(aD)
6
```

Go to Activity

Modifiers

- Modifiers in the *M269 Companion*

```
Python3>>> aD = {'to': 2, 'be': 2
...              , 'or': 1, 'not': 1}
Python3>>> aD
{'not': 1, 'be': 2, 'to': 2, 'or': 1}
Python3>>> aD['dobe'] = 2
Python3>>> aD
{'not': 1, 'be': 2, 'dobe': 2, 'to': 2, 'or': 1}
Python3>>> aD['do'] = 1
Python3>>> aD
{'do': 1, 'to': 2, 'or': 1
, 'not': 1, 'be': 2, 'dobe': 2}
Python3>>> aD['be'] = 3
Python3>>> aD
{'do': 1, 'to': 2, 'or': 1
, 'not': 1, 'be': 3, 'dobe': 2}
```

- Modifiers in the *M269 Companion*

```
Python3>>> aD
{'do': 1, 'to': 2, 'or': 1
, 'not': 1, 'be': 3, 'dobe': 2}
Python3>>> aD['do'] = aD['do'] + 1
Python3>>> aD
{'do': 2, 'to': 2, 'or': 1
, 'not': 1, 'be': 3, 'dobe': 2}
Python3>>> del aD['dobe']
Python3>>> aD
{'do': 2, 'to': 2, 'or': 1
, 'not': 1, 'be': 3}
```

- It is an error to access a non-existant key

```
Python3>>> aD
{'do': 2, 'to': 2, 'or': 1
, 'not': 1, 'be': 3}
Python3>>> aD['dobe']
Traceback (most recent call last):
```

```
  File "<stdin>", line 1, in <module>
KeyError: 'dobe'
Python3>>> del aD['dobe']
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
KeyError: 'dobe'
Python3>>> aD['dobe'] = aD['dobe'] + 1
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
KeyError: 'dobe'
```

- If a key occurs more than once, the last value for that key becomes the corresponding value in the new dictionary. Not an error but could catch you out

```
Python3>>> fD = {'one' : 2, 'one': 3}
Python3>>> fD
{'one': 3}
```

Add One Exercise

Activity 5 Add One Exercise

- Write a function, addOneToKey, that takes a key, key, a dictionary, xD, and adds 1 to the value of the key

Go to Answer

Answer 5 Add One Exercise

- Answer 5 Add One Exercise

```
def addOneToKey(key, xD) :
  """
  addOneToKey takes a key and a dictionary
  and adds one to the value of the key
  Invariant for xD:
    if key in xD :
      xD[key] > 0
  """

  if key in xD :
    xD[key] = xD[key] + 1
  else :
    xD[key] = 1

  return xD
```

Answer 5 Add One Exercise

- Answer 5 Add One Exercise
- Note that addOneToKey has a side effect on the input dictionary

```
Python3>>> aD
{'do': 2, 'to': 2, 'or': 1, 'not': 1, 'be': 3}
Python3>>> addOneToKey('do',aD)
{'do': 3, 'to': 2, 'or': 1, 'not': 1, 'be': 3}
Python3>>> aD
{'do': 3, 'to': 2, 'or': 1, 'not': 1, 'be': 3}
Python3>>> addOneToKey('abba',aD)
{'do': 3, 'abba': 1, 'to': 2, 'or': 1
, 'not': 1, 'be': 3}
Python3>>> aD
{'do': 3, 'abba': 1, 'to': 2, 'or': 1
, 'not': 1, 'be': 3}
```

Go to Activity

Delete One Exercise

Activity 6 Delete One Exercise

- Write a function, delOneFromKey, that takes a key, key, a dictionary, xD, and subtracts 1 from the value of the key if the key is in xD

Go to Answer

Answer 6 Delete One Exercise

- Answer 6 Delete One Exercise

```
def delOneFromKey(key, xD) :
  """
  delOneFromKey takes a key and a dictionary
  and deletes one from the value of the key
  Invariant for xD:
    if key in xD :
      xD[key] > 0
  """

  if not key in xD :
    pass
  elif xD[key] == 1 :
    del xD[key]
  else :
    xD[key] = xD[key] - 1

  return xD
```

Answer 6 Delete One Exercise

- Answer 6 Delete One Exercise — examples

```
Python3>>> aD
{'do': 3, 'abba': 1, 'to': 2, 'or': 1
, 'not': 1, 'be': 3}
Python3>>> delOneFromKey('do',aD)
{'do': 2, 'abba': 1, 'to': 2, 'or': 1
, 'not': 1, 'be': 3}
Python3>>> aD
{'do': 2, 'abba': 1, 'to': 2, 'or': 1
, 'not': 1, 'be': 3}
Python3>>> delOneFromKey('abba',aD)
{'do': 2, 'to': 2, 'or': 1, 'not': 1, 'be': 3}
Python3>>> aD
{'do': 2, 'to': 2, 'or': 1, 'not': 1, 'be': 3}
Python3>>> delOneFromKey('bbc',aD)
{'do': 2, 'to': 2, 'or': 1, 'not': 1, 'be': 3}
Python3>>> aD
{'do': 2, 'to': 2, 'or': 1, 'not': 1, 'be': 3}
```

Go to Activity

8 Abstract Data Types

8.1 Abstract Data Types — Overview

- **Abstract data type** is a type with associated operations, but whose representation is hidden (or not accessible)
- Common examples in most programming languages are Integer and Characters and other built in types such as tuples and lists

- Abstract data types are modeled on Algebraic structures
  - A set of values
  - Collection of operations on the values
  - Axioms for the operations may be specified as equations or pre and post conditions
- **Health Warning** different languages provide different ways of doing data abstraction with similar names that may mean subtly different things
- **Abstract Data Types and Object-Oriented Programming**
- Example: Shape with `Circle`s, `Square`s, ... and operations `draw`, `moveTo`, ...
- **ADT** approach centres on the data type — that tells you what shapes exist
- For each operation on shapes, you describe what they do for different shapes.
- **OO** you declare that to be a shape, you have to have some operations (`draw`, `moveTo`)
- For each kind of shape you provide an implementation of the operations
- **OO** easier to answer *What is a circle?* and add new shapes
- **ADT** easier to answer *How do you draw a shape?* and add new operations
- **Health Warning and Optional Material** Discussions about the merits of Functional programming and Object-oriented programming tend to look like the disputes between Lilliput and Blefuscu
- Abstract data type article contrasts ADT and OO as algebra compared to co-algebra
- What does *coalgebra* mean in the context of programming? is a fairly technical but accessible article.
- What does the *forall* keyword in Haskell do? — is an accessible article on *Existential Quantification*
- Bart Jacobs Coalgebra
- nLab Coalgebra
- Beware the distinction between *concepts* and *features* in programming languages — see OOP Disaster
- **Not for this session** — this slide is here just in case

```haskell
data Shape
  = Circle Point Radius
  | Square Point Size

draw :: Shape -> Pict
draw (Circle p r) = drawCircle p r
draw (Square p s) = drawRectangle p s s

moveTo :: Point -> Shape -> Shape
moveTo p2 (Circle p1 r) = Circle p2 r
moveTo p2 (Square p1 s) = Square p2 s

shapes :: [Shape]
shapes = [Circle (0,0) 1, Square (1,1) 2]

shapes01 :: [Shape]
shapes01 = map (moveTo (2,2)) shapes
```

- Example based on Lennart Augustsson email of 23 June 2005 on Haskell list

```haskell
class IsShape shape where
  draw :: shape -> Pict
  moveTo :: Point -> shape -> shape

data Shape = forall a . (IsShape a) => Shape a

data Circle = Circle Point Radius
instance IsShape Circle where
  draw (Circle p r) = drawCircle p r
  moveTo p2 (Circle p1 r) = Circle p2 r

data Square = Square Point Size
instance IsShape Square where
  draw (Square p s) = drawRectangle p s s
  moveTo p2 (Square p1 s) = Square p2 s

shapes :: [Shape]
shapes = [Shape (Circle (0,0) 10), Shape (Square (1,1) 2)]

shapes01 :: [Shape]
shapes01 = map (moveShapeTo (2,2)) shapes
           where
           moveShapeTo p (Shape s) = Shape (moveTo p s)
```

- Haskell Type Classes are similar to Java/OOP Interfaces
- See OOP vs type classes
- See Java's Interface and Haskell's type class: differences and similarities?
- See Difference between OOP interfaces and FP type classes
- See What exactly makes the Haskell type system so revered (vs say, Java)?
- **Health Warning** Much of OO programming is using the *OO syntax* to create ADTs

8.2 Marathon Example

- Marathon example from M269 2017J TMA02 Q 2 — description
- **Creator**
- `mthon = Marathon()` sets up a new marathon
- **Modifiers**
- `mthon.register(runner)` adds *runner* (unique string) to the marathon — called before race
- `mthon.finish(runner,time)` records finishing time (seconds) of *runner*. Rounding means several runners may have same time. Only called in the race via sensor on finish mat
- **Inspectors** only called after the race
- `mthon.place(runner)` returns place of the runner — places start at 1 (not 0) for shortest time
- `mthon.name(place)` returns the username of the runner in the given place.
- **Inspectors** called at any time
- `mthon.registered(runner)` returns `True` only if runner is registered

- `mthon.finished(runner)` returns `True` only if runner has finished the race
- `mthon.finishers()` returns the number of runners who have finished so far
- `mthon.finishersUpTo(time)` returns number of runners finished so far in given time (seconds) or less.

8.2.1 Marathon Example Questions

1. Describe how to assign different places to runners, even those with the same finishing times
2. State the preconditions for all functions excepts for `Marathon()`, `finishers()` and `finishersUpTo(time)`
3. Justify the data structures used and mention any alternatives considered
4. Implement the ADT
5. State the worst case complexity of each operation

- Which of the following are feasible patterns of `finish` calls ?

A.
```
mthon.finish('alice',180*60)
mthon.finish('bob',165*60)
mthon.finish('chris',185*60)
```

B.
```
mthon.finish('alice',180*60)
mthon.finish('bob',185*60)
mthon.finish('alice',185*60)
```

C.
```
mthon.finish('alice',180*60)
mthon.finish('bob',185*60)
mthon.finish('chris',185*60)
```

D.
```
mthon.finish('alice',180*60)
mthon.finish('chris',185*60)
mthon.finish('bob',185*60)
```

- Feasible patterns of `finish` calls ?
- Not feasible A (wrong order), B (duplicate entry)
- Feasible C, D (but *bob* and *chris* get different places)
- Hence assume the n^{th} call to `finish` is the n^{th} runner to finish — so same time is ordered by sequence of `finish` calls
- The function `finish` will allocate place *n* and the time in the *Marathon* data structure
- What should the preconditions to `finish` be ?
- Investigate preconditions by writing down the *type* of `finish`
- In pseudo-code (Haskell)

```
finish :: Marathon -> (Runner, Time) -> Marathon
```

- In words — `finish` takes a `Marathon` and a pair of `Runner` (String) and `Time` (Int minutes) and returns a `Marathon` (updated)

- Note we are abusing Haskell notation here since `finish` has a side effect on a marathon
- Write down the *types* of the remaining functions of the `Marathon`
- `register`
- `finish`
- `place`
- `name`
- `registered`
- `finished`
- `finishers`
- `finishersUpTo`
- Types of functions

```
register :: Marathon -> Runner -> Marathon

finish :: Marathon -> (Runner, Time) -> Marathon

place :: Marathon -> Runner -> Place

name :: Marathon -> Place -> Runner

registered :: Marathon -> Runner -> Bool

finished :: Marathon -> Runner -> Bool

finishers :: Marathon -> Int

finishersUpTo :: Marathon -> Time -> Int
```

- **Discussion** We have to choose our representation of `Marathon` and write the `__init__(self)` method
- `Lists` are mutable `sequences`
- `Tuples` are immutable `sequences`
- `Dictionaries` are mutable objects that can look up values by key (hashable name) such as a string
- With lists and tuples you can get out of range errors
- With dictionaries it is an error to look up a key that is not in the map
- Use the types to work out how we want to access the data and what choices we have to store the data
- What is the data type of `Runner` ?
- What is the data type of `Place` ?

Operation	Preconditions
register	not registered(*runner*)
finish	registered(*runner*) and *time* > 0
place	finished(*runner*)
name	0 < *place* <= finishers()
registered	True
finished	registered(*runner*)

8.2.2 Marathon Representation and Code

```
12 class Marathon :
13
14   # Creator
15   # -------
16
17   def __init__(self) :
18     """
19     Set up an empty Marathon
20     """
21     self.runnerToPlace = dict()
22     self.listOfRunnerTime = []
```

- runnerToPlace is a dictionary mapping *runner* to *place*
- Registered runners who have not finished have a place of 0
- listOfRunnerTime is a list of (*runner*, *time*) which maps *place* (list index + 1) to *runner* and *time*

```
81   # This modifier is called before the race starts
82   def register(self, runner) :
83     """
84     Register the runner
85     Returns nothing
86     Assumes the runner is not yet registered
87     """
88     self.runnerToPlace[runner] = 0
```

- runnerToPlace stores both *place* and whether registered

```
90   # This modifier is called during the race
91   def finish(self, runner, time) :
92     """
93     Record that the runner finished in given time
94     Return nothing
95     Assume the unit of time is seconds
96     Assume the runner has registered
97     """
98     place = self.finishers() + 1
99     self.listOfRunnerTime.append((runner, time))
100    self.runnerToPlace[runner] = place
```

- listOfRunnerTime gives the *place* by adding 1 to the list index
- This is equivalent to adding 1 to the number of runners who have finished so far

```
29   def registered(self, runner) :
30     """
31     Return True if runner has registered,
32     otherwise False
33     """
34     return runner in self.runnerToPlace
35
36   def finishers(self) :
37     """
```

```
38      Return number of finishers so far
39      """
40      return len(self.listOfRunnerTime)

42    def finished(self, runner) :
43      """
44      Return True if runner has finished
45      otherwise False
46      Assume the runner has registered
47      """
48      return self.runnerToPlace[runner] > 0

52    def finishersUpTo(self, time) :
53      """
54      Return how many runners finished
55      in given time or less
56      Assume the unit of time is seconds
57      """
58      for index in range(self.finishers()) :
59        if self.listOfRunnerTime[index][TIMEINDEX] > time :
60          return index
61      # All starting finished up to given time
62      return self.finishers()

64    def place(self, runner) :
65      """
66      Return place the runner finished
67      Assume runner finished
68      """
69      return self.runnerToPlace[runner]

71    def name(self, place) :
72      """
73      Return name of runner in given place
74      Assume place corresponds to a finisher
75      """
76      return self.listOfRunnerTime[place - 1][NAMEINDEX]
```

9 Search Trees

9.1 Height Balanced Trees

- See the slides, notes and examples in M269 Tutorial Binary Trees
- The section on AVL Trees gives the basic transformations
- The section on *Implementing Sets in AVL Trees* gives an application of AVL Trees

10 Graph Algorithms

10.1 Using the M269 Library

- Download the M269 Library from https://mwermelinger.github.io/m269-library/
- Copy the folders lib, doc, examples, help, tests to your working folder — I am using UsingM269Library
- __init__.py

- bst.py,deque.py,digraph.py, graph.py
- hash_table.py, priority_queue.py, queue.py, sort.py, stack.py
- Python Packages Documentation
- What is __init__.py for?
- Python Packaging Tutorials

bmrFrm

UsingM269Library.py

```
#!/usr/bin/env python3

from lib.graph import *

# Sample M269 Library graphs

pwrGr = Graph()

pwrGr.add_edge(1,  2, 100)
pwrGr.add_edge(2,  3, 100)
pwrGr.add_edge(3,  4,  50)
pwrGr.add_edge(3,  5,  50)
pwrGr.add_edge(6,  2,  50)
pwrGr.add_edge(6,  7,  50)
pwrGr.add_edge(7,  8,  40)
pwrGr.add_edge(8,  9,  10)
pwrGr.add_edge(8, 10,  10)
pwrGr.add_edge(8, 11,  20)
```

Using the M269 Library

```
<UsingM269Library><108> python3
Python3>>> from UsingM269Library import *
Python3>>> pwrGr.nodes()
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
Python3>>> type(pwrGr.nodes())
<class 'set'>
Python3>>> pwrGr.visited_dfs(2)
[2, 6, 7, 8, 11, 10, 9, 3, 5, 4, 1]
Python3>>> type(pwrGr.visited_dfs(2))
<class 'list'>
Python3>>> pwrGr.visited_dfs(2) == pwrGr.nodes()
False
Python3>>> set(pwrGr.visited_dfs(2))
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
Python3>>> pwrGr.visited_dfs(2) == len(pwrGr.nodes())
False
Python3>>> len(pwrGr.visited_dfs(2)) == len(pwrGr.nodes())
True
Python3>>> set(pwrGr.visited_dfs(2)) == pwrGr.nodes()
True
Python3>>>
```

10.1.1 Comparisons in Python

- Notice in the above examples comparing a list with set returned False, and comparing a list with the size (len) of a set returned False
- If you had previously used strongly typed languages you might have expected a TypeError

- Python Library: Comparisons says: *Objects of different types, except different numeric types, never compare equal.*
- Function objects always compare unequal (even though comparing functions for equality is in general not computable — see Unit 7)
- The last two examples give correct ways of comparing — though it would be best to do set equality comparison in this case.
- This must be a big source of bugs (and a shock to people like me)

Rationale for Equality Comparison across Types

- Python is weakly typed
- Dictionaries can have keys of mixed types — for example, integer `1` and string `'1'`
- See Python: Why does equality comparing an int with a string not throw an error?
- See Python: dict

10.1.2 The import Statement and Modules

- Python Language: The `import` statement is the reference for the `import` statement

```
from lib.graph import *
```

- All public names in `graph.py` are bound in the local namespace and we can refer to `Graph()`

```
import lib.graph
```

- `lib.graph` imported and we have to refer to `lib.graph.Graph()`
- Python Tutorial: Modules are described in the Python Tutorial — notice section 6.4.1 *Importing * from a Package*
- Python Command Line: PYTHONPATH can augment the default search path for module files
- The Python search path can be manipulated with sys.path

11 Greedy Algorithms

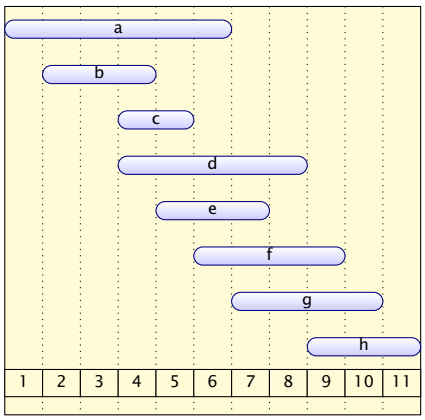
11.1 Interval Scheduling

- Greedy algorithms follow the problem solving heuristic of making the locally optimal choice at each stage with the intent of finding a global optimum
- In general this rarely works — but it does in some cases including
  - Dijkstra's algorithm and A* search algorithm for graph search and shortest path finding

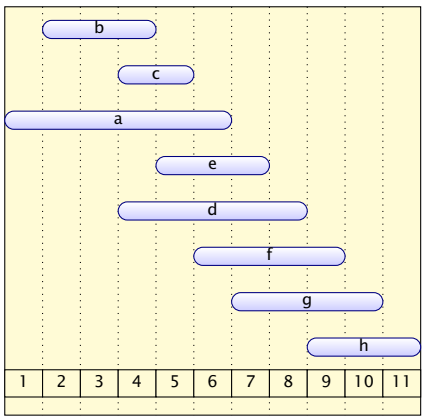
  - Kruskal's algorithm and Prim's algorithm for constructing minimum spanning trees of a given connected graph
  - Interval scheduling or Activity selection problem to find the maximum number of activities that do not clash with each other

- If a greedy algorithm can be proven to yield the global optimum for a given problem class, it typically becomes the method of choice because it is faster than other optimization methods such as dynamic programming.
- Interval scheduling
- Job j starts at s_j and finishes at f_j
- Two jobs are compatible if they do not overlap
- **Q** What is the maximum subset of mutually compatible jobs?
- **Greedy template** Consider jobs in some order. Take each job provided it is compatible with the ones already taken.
- **Exercise** What orderings can we have ?
- Example from Greedy algorithms: Interval scheduling
- **Greedy template** Consider jobs in some order. Take each job provided it is compatible with the ones already taken.
- **Earliest start time** Consider jobs in ascending order of start time s_j
- **Earliest finish time** Consider jobs in ascending order of finish time f_j
- **Shortest interval** Consider jobs in ascending order of interval length $f_j + 1 - s_j$
- **Fewest conflicts** For each job, count the number of conflicting jobs c_j and schedule in ascending order of conflicts c_j
- For the jobs given below, produce an ordering by each of the greedy templates (above) and the schedule produced
- Each triple in the list below means $(\texttt{name}, s_i, f_i)$ where the times are inclusive

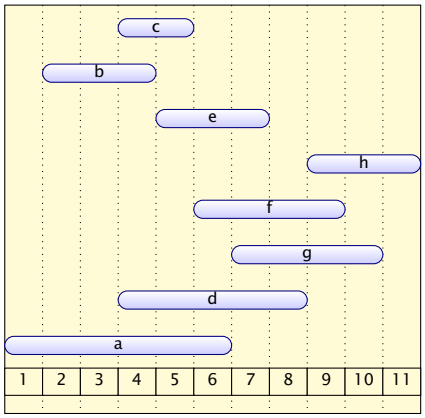
```
jobs
 = [(a,1,6),(b,2,4),(c,4,5),(d,4,8)
   ,(e,5,7),(f,6,9),(g,7,10),(h,9,11)]
```



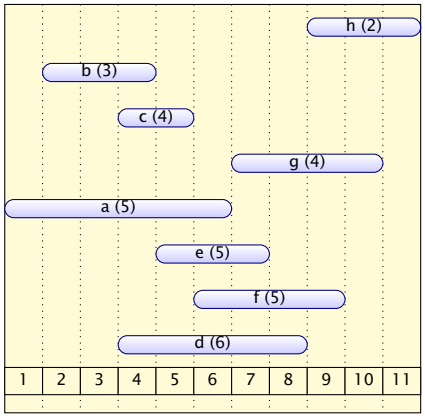
- Schedule jobs **a**, **g** (2 jobs)



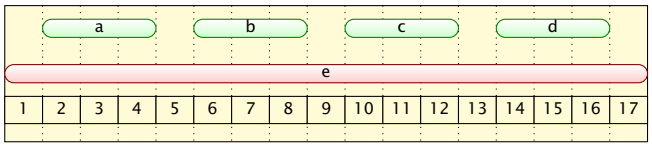
- Schedule jobs **b**, **e**, **h** (3 jobs)



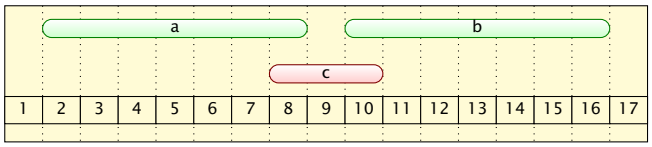
- Schedule jobs **c**, **h** (2 jobs)



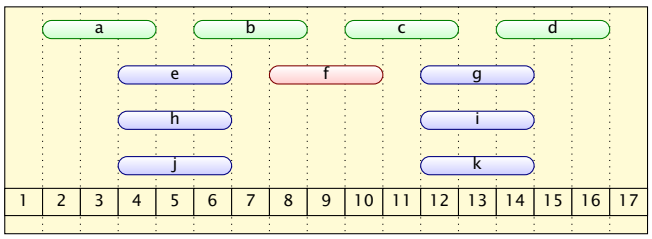
- Schedule jobs **h**, **b**, **e** (3 jobs)
- For each of the following *Greedy Templates* produce a counter example to show it may not produce the optimal schedule
- **Earliest start time** Consider jobs in ascending order of start time s_j
- **Shortest interval** Consider jobs in ascending order of interval length $f_j + 1 - s_j$
- **Fewest conflicts** For each job, count the number of conflicting jobs c_j and schedule in ascending order of conflicts c_j



- **e** dominates the optimal schedule by starting earlier and overlapping the others



- **c** dominates the optimal schedule y being shorter and overlapping the other two



- **f** dominates the optimal schedule by only having two conflicts and overlapping **b** and **c**

Order by Earliest Finish Time — Optimality Proof

- Basic structure of correctness proof:
- Assume that there is an optimal solution that is different from the greedy solution.
- Find the *first* difference between the two solutions.
- Argue that we can exchange the optimal choice for the greedy choice without making the solution worse (although the exchange might not make it better).
- This argument implies by induction that some optimal solution *contains* the entire greedy solution, and therefore *equals* the greedy solution.
- Sometimes, an additional step is required to show no optimal solution *strictly* improves the greedy solution.
- See Jeff Erickson: Algorithms
- Proof also in Interval Scheduling and Greedy Algorithms
- The slides at Kevin Wayne: Greedy Algorithms are from Kleinberg and Éva Tardos (2013)

12 Dynamic Programming

12.1 Edit Distance

- See the slides, notes and examples in M269 Tutorial Graph Algorithms
- The section on Dynamic Programming gives the basic formulation
- The section on *Edit Distance* gives an application

13 Future Work

- Wednesday, 24 March 2021 TMA02 due
- Sunday, 11 April 2021 online tutorial Unit 6 Logic
- Wednesday 28 April 2021 iCMA46 due
- Sunday, 3 May 2020 online tutorial Unit 7 Computability, Complexity
- Sunday, 16 May 2021 online tutorial exam revision
- Saturday, 22 May 2021 tutorial online exam revision
- Tuesday 25 May 2021 iCMA47 due
- Tuesday 8 June 2021 Exam
- Please email me with any requests for particular topics

14 Web Sites & References

14.1 Sorting

- **Big-O Cheat Sheet** http://bigocheatsheet.com/

14.2 Graph Algorithms

- **Graph Theory** http://en.wikipedia.org/wiki/Graph_theory
- **Tree (Graph Theory)** http://en.wikipedia.org/wiki/Tree_(graph_theory)
- **Dekai Wu Algorithms course** https://www.cse.ust.hk/~dekai/271/
- **Jeff Erickson Algorithms** http://jeffe.cs.illinois.edu/teaching/algorithms/

Web Sites for Dynamic Programming

- *Jeff Erickson Algorithms* jeffe.cs.illinois.edu/teaching/algorithms
- Cormen et al. (2009, page 407) has same example as *Jeff Erickson*
- *Peter Norvig Spell Checker* norvig.com/spell-correct.html (from stackoverflow.com/questions/2 distance-in-python)
- *Rosetta Code: Levenshtein Distance* rosettacode.org/wiki/Levenshtein_distance
- *Edit Distance HaskellWiki* https://wiki.haskell.org/Edit_distance
- *Edit Distance in Haskell* stackoverflow.com/questions/5515025/edit-distance-algorithm-in-haskell-performance-tuning
- *Wikibooks Algorithm implementation — Levenshtein Distance* en.wikibooks.org/wiki/Algorithm_ Implementation/Strings/Levenshtein_distance

- *Wikipedia Levenshtein Distance* en.wikipedia.org/wiki/Levenshtein_distance
- *Needleman-Wunsch algorithm* en.wikipedia.org/wiki/Needleman-Wunsch_algorithm

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