Computability, Complexity

M269 Unit 7

Phil Molyneux

2 May 2021

M269 Unit 7

Computability, Complexity Tutorial

- Welcome & Introductions
- Computability topics:
  - Ideas of Computation and Algorithms
  - Problem Reduction
  - Turing Machines
  - Undecidable, Semi-decidable and decidable problems
  - Effective Computability: Turing machines, Lambda Calculus, μ-recursive functions
  - *Optional topic* Lambda Calculus introduction
- Complexity topics
- Exercises similar to CMAs and exam
- *Key aim:* Identify where people have problems and how to overcome them.
- *Adobe Connect* — if you or I get cut off, wait till we reconnect (or send you an email)
- **Recording** Meeting > Record Meeting... ✔

M269 Tutorial

Introductions — Me

- *Name* Phil Molyneux
- *Background* Physics and Maths, Operational Research, Computer Science
- *First programming languages* Fortran, BASIC, Pascal
- *Favourite Software*
  - Haskell — pure functional programming language
  - Text editors TextMate, Sublime Text — previously Emacs
  - Word processing and presentation slides in LaTeX
  - Mac OS X
- *Learning style* — I read the manual before using the software (really)

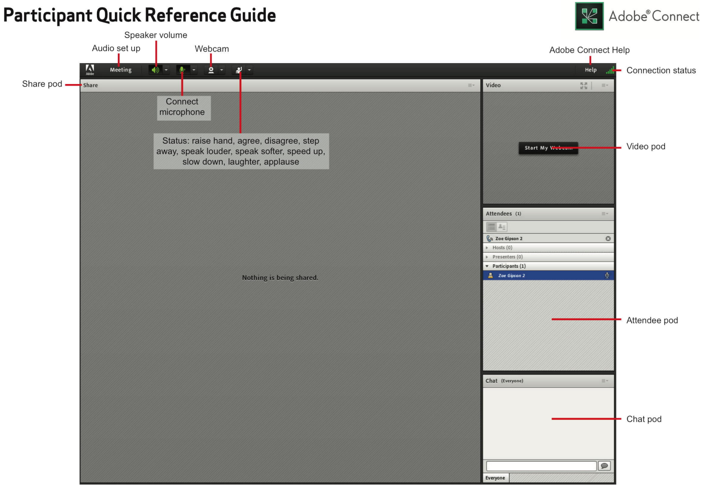
M269 Tutorial

Introductions — You

- *Name* ?
- *Position in M269* ? Which part of which Units and/or Reader have you read ?
- *Particular topics you want to look at* ?
- *Learning Syle* ?

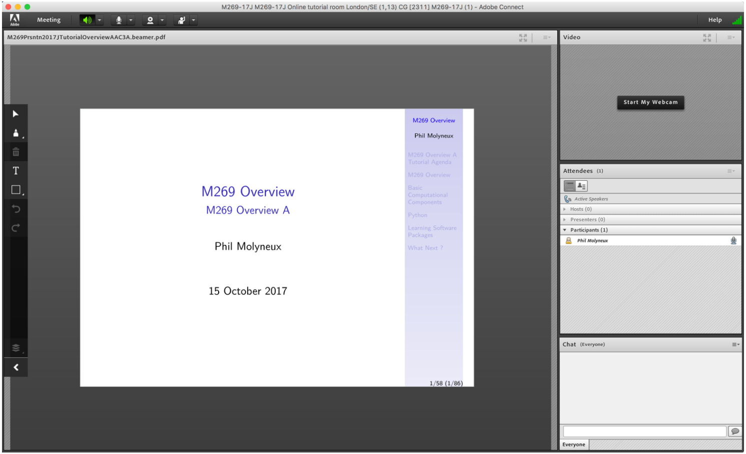
Adobe Connect

Interface — Student Quick Reference



Adobe Connect

Interface — Student View



Adobe Connect

Settings

- **Everybody: Audio Settings** Meeting › Audio Setup Wizard. . .
- **Audio** Menu bar › Audio › Microphone rights for Participants ✔
- Do not *Enable single speaker mode*
- **Drawing Tools** Share pod menu bar › Draw (1 slide/screen)
- Share pod menu bar › Menu icon › Enable Participants to draw ✔ gray
- Meeting › Preferences › Whiteboard › Enable Participants to draw ✔
- Cancel hand tool . . . Do not enable green pointer. . .
- Meeting › Preferences › Attendees Pod ✘ *Raise Hand notification*
- Meeting › Preferences › Display Name Display First & Last Name
- **Cursor** Meeting › Preferences › General tab › Host Cursors › Show to all attendees ✔ (default *Off*)
- Meeting › Preferences › Screen Share › Cursor › Show Application Cursor
- **Webcam** Menu bar › Webcam › Enable Webcam for Participants ✔
- **Recording** Meeting › Record Meeting. . . ✔

Adobe Connect

Access

- **Tutor Access**

  TutorHome > M269 Website > Tutorials

  Cluster Tutorials > M269 Online tutorial room

  Tutor Groups > M269 Online tutor group room

  Module-wide Tutorials > M269 Online module-wide room

- **Attendance**

  TutorHome > Students > View your tutorial timetables

- **Beamer Slide Scaling** 440% (422 x 563 mm)
- **Clear Everyone's Status**

  Attendee Pod > Menu > Clear Everyone's Status

- **Grant Access** and send link via email

  Meeting > Manage Access & Entry > Invite Participants...

- **Presenter Only Area**

  Meeting > Enable/Disable Presenter Only Area

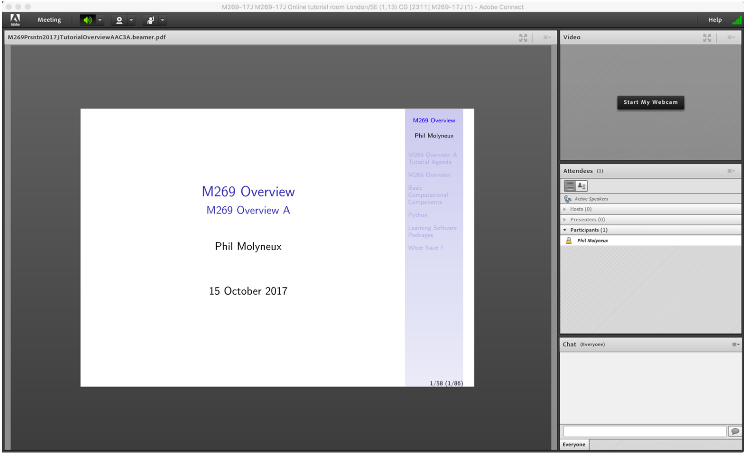
Adobe Connect

Keystroke Shortcuts

- Keyboard shortcuts in Adobe Connect
- **Toggle Mic** ⌘ + M (Mac), Ctrl + M (Win) (On/Disconnect)
- **Toggle Raise-Hand status** ⌘ + E
- **Close dialog box** ⎋ (Mac), Esc (Win)
- **End meeting** ⌘ + \

Adobe Connect Interface

Student View (default)

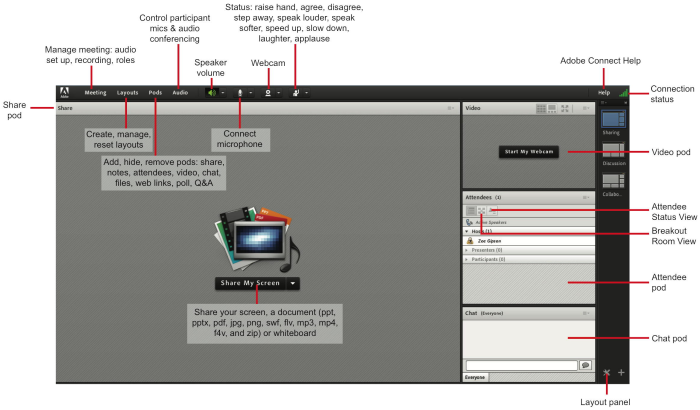


Adobe Connect Interface

Tutor View

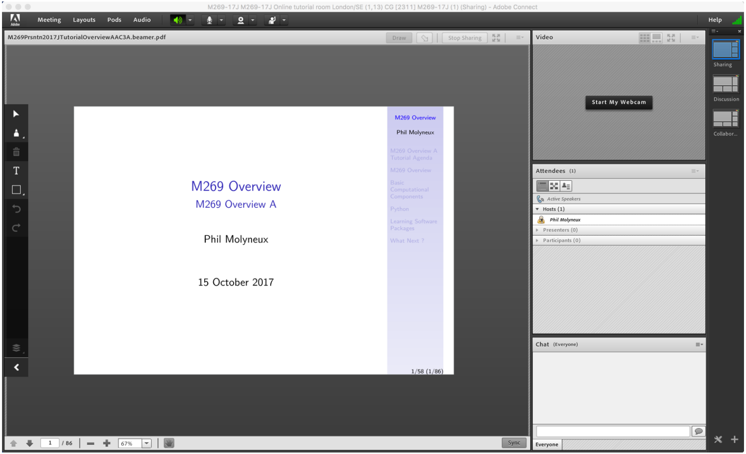
Host Quick Reference Guide

Adobe® Connect



Adobe Connect Interface

Tutor View



Adobe Connect Interface

Sharing Screen & Applications

- Share My Screen ❯ Application tab ❯ Terminal for Terminal
- Share menu ❯ Change View ❯ Zoom in for mismatch of screen size/resolution (Participants)
- (Presenter) Change to 75% and back to 100% (solves participants with smaller screen image overlap)
- Leave the application on the original display
- Beware blued hatched rectangles — from other (hidden) windows or contextual menus
- Presenter screen pointer affects viewer display — beware of moving the pointer away from the application
- First time: System Preferences ❯ Security & Privacy ❯ Privacy ❯ Accessibility

Adobe Connect

Ending a Meeting

- *Notes for the tutor only*
- **Student:** Meeting › Exit Adobe Connect
- **Tutor:**
- **Recording** Meeting › Stop Recording ✔
- **Remove Participants** Meeting › End Meeting. . . ✔
  - Dialog box allows for message with default message:
  - *The host has ended this meeting. Thank you for attending.*
- **Recording availability** *In course Web site for joining the room, click on the eye icon in the list of recordings under your recording* — edit description and name
- **Meeting Information** Meeting › Manage Meeting Information — can access a range of information in Web page.
- **Attendance Report** see course Web site for joining room

Adobe Connect

Invite Attendees

- **Provide Meeting URL** Menu › Meeting › Manage Access & Entry › Invite Participants. . .
- **Allow Access without Dialog** Menu › Meeting › Manage Meeting Information provides new browser window with *Meeting Information* Tab bar › Edit Information
- Check *Anyone who has the URL for the meeting can enter the room*
- Default *Only registered users and accepted guests may enter the room*
- **Reverts to default next session but URL is fixed**
- Guests have blue icon top, registered participants have yellow icon top — same icon if URL is open
- See Start, attend, and manage Adobe Connect meetings and sessions

Adobe Connect

Layouts

- **Creating new layouts** example *Sharing* layout
- Menu › Layouts › Create New Layout. . . Create a New Layout dialog › Create a new blank layout and name it *PMolyMain*
- New layout has no Pods but does have Layouts Bar open (see Layouts menu)
- **Pods**
- Menu › Pods › Share › Add New Share and resize/position — initial name is *Share n*
- **Rename Pod** Menu › Pods › Manage Pods. . . Manage Pods › Select › Rename or Double-click & rename
- Add Video pod and resize/reposition
- Add Attendance pod and resize/reposition
- Add Chat pod — name it *PMolyChat* — and resize/reposition

Adobe Connect

Layouts

- Dimensions of **Sharing** layout (on 27-inch iMac)
  - Width of Video, Attendees, Chat column 14 cm
  - Height of Video pod 9 cm
  - Height of Attendees pod 12 cm
  - Height of Chat pod 8 cm
- **Duplicating Layouts** does *not* give new instances of the Pods and is probably not a good idea (apart from local use to avoid delay in reloading Pods)

Adobe Connect

Chat Pods

- **Format Chat text**
- Chat Pod › menu icon › My Chat Color
- Choices: Red, Orange, Green, Brown, Purple, Pink, Blue, Black
- Note: Color reverts to Black if you switch layouts
- Chat Pod › menu icon › Show Timestamps

Computability

Ideas of Computation

- The idea of an algorithm and what is effectively computable
- **Church-Turing thesis** Every function that would naturally be regarded as computable can be computed by a deterministic Turing Machine. (Unit 7 Section 4)
- See Phil Wadler on computability theory performed as part of the Bright Club at The Strand in Edinburgh, Tuesday 28 April 2015

Computability

Models of Computation

- In automata theory, a *problem* is the question of deciding whether a given string is a member of some particular language
- If Σ is an alphabet, and L is a language over Σ, that is $L \subseteq \Sigma^*$, where Σ^* is the set of all strings over the alphabet Σ then we have a more formal definition of *decision problem*
- Given a string $w \in \Sigma^*$, decide whether $w \in L$
- Example: Testing for a prime number — can be expressed as the language L_p consisting of all binary strings whose value as a binary number is a prime number (only divisible by 1 or itself)
- See Hopcroft (2007, section 1.5.4)

Automate Theory

Alphabets, Strings

- An **Alphabet**, Σ, is a finite, non-empty set of symbols.
- Binary alphabet $\Sigma = \{0, 1\}$
- Lower case letters $\Sigma = \{a, b, \ldots, z\}$
- A **String** is a finite sequence of symbols from some alphabet
- 01101 is a string from the Binary alphabet $\Sigma = \{0, 1\}$
- The **Empty string**, ϵ, contains no symbols
- **Powers**: Σ^k is the set of strings of length k with symbols from Σ
- The set of all strings over an alphabet Σ is denoted Σ^*
- $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$
- **Question** Does $\Sigma^0 = \emptyset$? ($\emptyset$ is the empty set)

Automata Theory

Languages

- An **Language**, L, is a subset of Σ^*
- The set of binary numerals whose value is a prime
  $\{10, 11, 101, 111, 1011, \ldots\}$
- The set of binary numerals whose value is a square
  $\{100, 1001, 10000, 11001, \ldots\}$

Computability

Church-Turing Thesis & Quantum Computing

- **Church-Turing thesis** Every function that would naturally be regarded as computable can be computed by a deterministic Turing Machine.
- **physical Church-Turing thesis** Any finite physical system can be simulated (to any degree of approximation) by a Universal Turing Machine.
- **strong Church-Turing thesis** Any finite physical system can be simulated (to any degree of approximation) with polynomial slowdown by a Universal Turing Machine.
- Shor's algorithm (1994) — quantum algorithm for factoring integers — an NP problem that is not known to be P — also not known to be NP-complete and we have no proof that it is not in P
- Reference: Section 4 of Unit 6 & 7 Reader

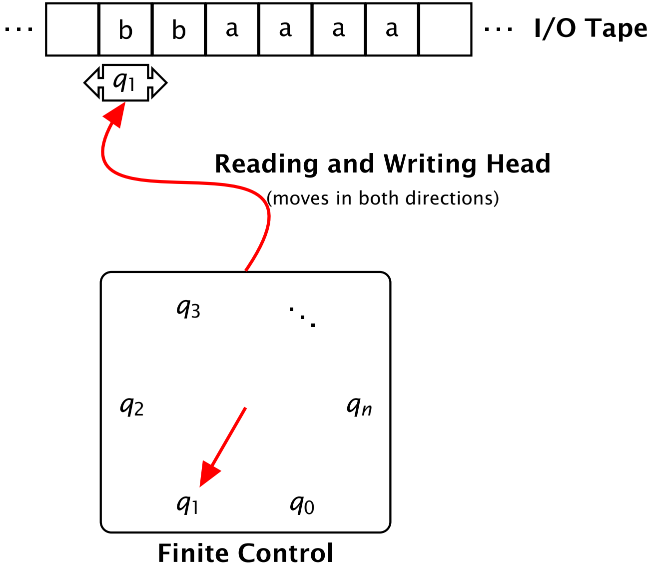
Computability

Turing Machine

- **Finite control** which can be in any of a finite number of *states*
- **Tape** divided into cells, each of which can hold one of a finite number of symbols
- Initially, the **input**, which is a finite-length string of symbols in the *input alphabet*, is placed on the tape
- All other tape cells (extending unbounded left and right) hold a special symbol called *blank*
- A **tape head** which initially is over the leftmost input symbol
- A **move** of the Turing Machine depends on the state and the tape symbol scanned
- A move can change state, write a symbol in the current cell, move left, right or stay
- References: Hopcroft (2007, page 326), Unit 6 & 7 Reader (section 5.3)

Turing Machine Diagram

Turing Machine Diagram



Computability

Turing Machine notation

- Q finite set of states of the finite control
- Σ finite set of *input symbols* (M269 S)
- Γ complete set of *tape symbols* $\Sigma \subset \Gamma$
- δ Transition function (M269 instructions, I)
  $\delta :: Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$
  $\delta(q, X) \mapsto (p, Y, D)$
- $\delta(q, X)$ takes a state, q and a tape symbol, X and returns (p, Y, D) where p is a state, Y is a tape symbol to overwrite the current cell, D is a direction, Left, Right or Stay
- q_0 *start state* $q_0 \in Q$
- B *blank symbol* $B \in \Gamma$ and $B \notin \Sigma$
- F set of *final or accepting states* $F \subseteq Q$

Turing Machine Examples

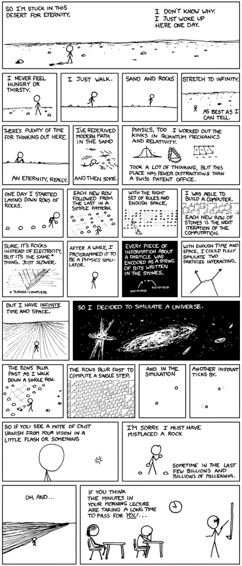
Turing Machine Simulators

- Morphett's Turing machine simulator — the examples below are adapted from here
- Ugarte's Turing machine simulator
- XKCD A Bunch of Rocks — XKCD Explanation

  Image below (will need expanding to be readable)

Turing Machine Examples

XKCD A Bunch of Rocks



Turing Machine Examples

The Successor Function

- **Input** binary representation of numeral n
- **Output** binary representation of $n + 1$
- Example $1010 \mapsto 1011$ and $1011 \mapsto 1100$
- Initial cell: leftmost symbol of n
- **Strategy**
- **Stage A** make the rightmost cell the current cell
- **Stage B** Add 1 to the current cell.
- If the current cell is 0 then replace it with 1 and go to stage C
- If the current cell is 1 replace it with 0 and go to stage B and move Left
- If the current cell is blank, replace it by 1 and go to stage C
- **Stage C** Finish up by making the leftmost cell current

Turing Machine Examples

The Successor Function (2)

- Represent the Turing Machine program as a list of quintuples (q, X, p, Y, D)
- **Stage A**

  $(q_0, 0, q_0, 0, R)$

  $(q_0, 1, q_0, 1, R)$

  (q_0, B, q_1, B, L)
- **Stage B**

  $(q_1, 0, q_2, 1, S)$

  $(q_1, 1, q_1, 0, L)$

  $(q_1, B, q_2, 1, S)$
- **Stage C**

  $(q_2, 0, q_2, 0, L)$

  $(q_2, 1, q_2, 1, L)$

  (q_2, B, q_h, B, R)

Turing Machine Examples

The Successor Function (2a)

- **Exercise** Translate the quintuples (q, X, p, Y, D) into English and check they are the same as the specification
- **Stage A** make the rightmost cell the current cell

  $(q_0, 0, q_0, 0, R)$

  If state q_0 and read symbol 0 then stay in state q_0 write 0, move R

  $(q_0, 1, q_0, 1, R)$

  If state q_0 and read symbol 1 then stay in state q_0 write 1, move R

  (q_0, B, q_1, B, L)

  If state q_0 and read symbol B then state q_1 write B, move L

Turing Machine Examples

The Successor Function (2b)

- **Exercise** Translate the quintuples (q, X, p, Y, D) into English
- **Stage B** Add 1 to the current cell.

  $(q_1, 0, q_2, 1, S)$

  If state q_1 and read symbol 0 then state q_2 write 1, stay

  $(q_1, 1, q_1, 0, L)$

  If state q_1 and read symbol 1 then state q_1 write 0, move L

  $(q_1, B, q_2, 1, S)$

  If state q_1 and read symbol B then state q_2 write 1, stay

Turing Machine Examples

The Successor Function (2c)

- **Exercise** Translate the quintuples (q, X, p, Y, D) into English
- **Stage C** Finish up by making the leftmost cell current

  $(q_2, 0, q_2, 0, L)$

  If state q_2 and read symbol 0 then state q_2 write 0, move L

  $(q_2, 1, q_2, 1, L)$

  If state q_2 and read symbol 1 then state q_2 write 0, move L

  (q_2, B, q_h, B, R)

  If state q_2 and read symbol B then state q_h write B, move R HALT
- Notice that the Turing Machine feels like a series of `if ... then` or `case` statements inside a `while` loop

Turing Machine Examples

The Successor Function (3)

- **Sample Evaluation** $11 \mapsto 100$
- **Representation** $\cdots BX_1 X_2 \cdots X_{i-1}\, q X_i X_{i+1} \cdots X_n B \cdots$

  $q_0 11$

  $1 q_0 1$

  $11 q_0 B$

  $1 q_1 1$

  $q_1 10$

  $q_1 B00$

  $q_2 100$

  $q_2 B100$

  $q_h 100$
- **Exercise** evaluate $1011 \mapsto 1100$

Turing Machine Examples

Instantaneous Description

- **Representation** $\cdots BX_1X_2\cdots X_{i-1}qX_iX_{i+1}\cdots X_nB\cdots$
- q is the state of the TM
- The head is scanning the symbol X_i
- Leading or trailing blanks B are (usually) not shown unless the head is scanning them
- $\vdash_M$ denotes one move of the TM M
- $\vdash_M^*$ denotes zero or more moves
- $\vdash$ will be used if the TM M is understood
- If (q, X_i, p, Y, L) denotes a TM move then $X_1\cdots X_{i-1}qX_i\cdots X_n \vdash_M X_1\cdots X_{i-2}pX_{i-1}Y\cdots X_n$

Turing Machine Examples

The Binary Palindrome Function

- **Input** binary string s
- **Output** YES if palindrome, NO otherwise
- Example 1010 $\mapsto$ *NO* and 1001 $\mapsto$ *YES*
- Initial cell: leftmost symbol of s
- **Strategy**
- **Stage A** read the leftmost symbol
- If blank then accept it and go to stage D otherwise erase it
- **Stage B** find the rightmost symbol
- If the current cell matches leftmost recently read then erase it and go to stage C
- Otherwise reject it and go to stage E
- **Stage C** return to the leftmost symbol and stage A
- **Stage D** print YES and halt
- **Stage E** erase the remaining string and print NO

Turing Machine Examples

The Binary Palindrome Function (2)

- Represent the Turing Machine program as a list of quintuples (q, X, p, Y, D)
- **Stage A** read the leftmost symbol

  $(q_0, 0, q_{1_o}, B, R)$

  $(q_0, 1, q_{1_i}, B, R)$

  (q_0, B, q_5, B, S)
- **Stage B** find rightmost symbol

  $(q_{1_o}, B, q_{2_o}, B, L)$

  $(q_{1_o}, *, q_{1_o}, *, R)$ * is a wild card, matches anything

  $(q_{1_i}, B, q_{2_i}, B, L)$

  $(q_{1_i}, *, q_{1_i}, *, R)$

Turing Machine Examples

The Binary Palindrome Function (3)

- **Stage B** check

  $(q_{2_o}, 0, q_3, B, L)$

  (q_{2_o}, B, q_5, B, S)

  $(q_{2_o}, *, q_6, *, S)$

  $(q_{2_i}, 1, q_3, B, L)$

  (q_{2_i}, B, q_5, B, S)

  $(q_{2_i}, *, q_6, *, S)$

- **Stage C** return to the leftmost symbol and stage A

  (q_3, B, q_5, B, S)

  $(q_3, *, q_4, *, L)$

  (q_4, B, q_0, B, R)

  $(q_4, *, q_4, *, L)$

Turing Machine Examples

The Binary Palindrome Function (4)

- **Stage D** accept and print YES

  $(q_5, *, q_{5_a}, Y, R)$

  $(q_{5_a}, *, q_{5_b}, E, R)$

  $(q_{5_b}, *, q_7, S, S)$

- **Stage E** erase the remaining string and print NO

  (q_6, B, q_{6_a}, N, R)

  $(q_6, *, q_6, B, L)$

  $(q_{6_a}, *, q_7, O, S)$

- Finish

  (q_7, B, q_h, B, R)

  $(q_7, *, q_7, *, L)$

Turing Machine Examples

The Binary Palindrome Function (4)

- **Sample Evaluation** $101 \mapsto \text{YES}$

  $q_0 101 \vdash Bq_{1_i}01 \vdash B0q_{1_i}1 \vdash B01q_{1_i}B$

  $\vdash B0q_{2_i}1$

  $\vdash Bq_3 0B \vdash q_4 B0B$

  $\vdash Bq_0 0B \vdash BBq_{1_o}B$

  $\vdash Bq_{2_o}BB$

  $\vdash Bq_5 BB \vdash Yq_{5_a}B \vdash YEq_{5_b}B \vdash YEq_7 S$

  $\vdash Yq_7 ES \vdash Bq_7 YES \vdash q_7 BYES \vdash q_h YES$
- **Exercise** Evaluate $110 \mapsto \text{NO}$

Turing Machine Examples

Binary Addition Example

- **Input** two binary numerals separated by a single space $n1 \ n2$
- **Output** binary numeral which is the sum of the inputs
- Example $110110 + 101011 \mapsto 1100001$
- Initial cell: leftmost symbol of $n1 \ n2$
- **Insight** look at the arithmetic algorithm

$$
\begin{array}{ccccccc}
\sqcup & 1 & 1 & 0 & 1 & 1 & 0 \\
\sqcup & 1 & 0 & 1 & 0 & 1 & 1 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

- **Discussion** how can we overwrite the first number with the result and remember how far we have gone ?

Turing Machine Examples

Binary Addition Example — Arithmetic Reinvented

␣	1	1	0	1	1	0
␣	1	0	1	0	1	1
␣	1	1	0	1	1	y
␣	1	0	1	0	1	␣
␣	1	1	1	0	x	y
␣	1	0	1	0	␣	␣
␣	1	1	1	x	x	y
␣	1	0	1	␣	␣	␣
1	0	0	x	x	x	y
␣	1	0	␣	␣	␣	␣
1	0	x	x	x	x	y
␣	1	␣	␣	␣	␣	␣
1	y	x	x	x	x	y
␣	␣	␣	␣	␣	␣	␣
1	1	0	0	0	0	1

Turing Machine Examples

Binary Addition Example (2)

- **Input** two binary numerals separated by a single space $n1\ n2$
- **Output** binary numeral which is the sum of the inputs
- Example $110110 + 101011 \mapsto 1100001$
- Initial cell: leftmost symbol of $n1\ n2$
- **Strategy**
- **Stage A** find the rightmost symbol

  If the symbol is 0 erase and go to stage Bx

  If the symbol is 1 erase go to stage By

  If the symbol is blank go to stage F

  dealing with each digit in $n1$

  if no further digits in $n1$ go to final stage
- **Stage Bx** Move left to a blank go to stage Cx
- **Stage By** Move left to a blank go to stage Cy

  moving to $n1$

Turing Machine Examples

Binary Addition Example (3)

- **Stage Cx** Move left to find first 0, 1 or B

  Turn 0 or B to X, turn 1 to Y and go to stage A

  adding 0 to a digit finalises the result (no *carry one*)
- **Stage Cy** Move left to find first 0, 1 or B

  Turn 0 or B to 1 and go to stage D

  Turn 1 to 0, move left and go to stage Cy

  dealing with the *carry one* in school arithmetic
- **Stage D** move right to X, Y or B and go to stage E
- **Stage E** replace 0 by X, 1 by Y, move right and go to Stage A

  finalising the value of a digit resulting from a *carry*
- **Stage F** move left and replace X by 0, Y by 1 and at B halt

Turing Machine Examples

Binary Addition Example (4)

- Represent the Turing Machine program as a list of quintuples (q, X, p, Y, D)
- **Stage A** find the rightmost symbol

  (q_0, B, q_1, B, R)

  $(q_0, *, q_0, *, R)$ * is a wild card, matches anything

  (q_1, B, q_2, B, L)

  $(q_1, *, q_1, *, R)$

  $(q_2, 0, q_{3_x}, B, L)$

  $(q_2, 1, q_{3_y}, B, L)$

  (q_2, B, q_7, B, L)

Turing Machine Examples

Binary Addition Example (5)

- **Stage Bx** move left to blank

  $(q_{3_x}, B, q_{4_x}, B, L)$

  $(q_{3_x}, *, q_{3_x}, *, L)$

- **Stage By** move left to blank

  $(q_{3_y}, B, q_{4_y}, B, L)$

  $(q_{3_y}, *, q_{3_y}, *, L)$

- **Stage Cx** move left to 0, 1, or blank

  $(q_{4_x}, 0, q_0, x, R)$

  $(q_{4_x}, 1, q_0, y, R)$

  (q_{4_x}, B, q_0, x, R)

  $(q_{4_x}, *, q_{4_x}, *, L)$

Turing Machine Examples

Binary Addition Example (6)

- **Stage Cy** move left to 0, 1, or blank

  $(q_{4_y}, 0, q_5, 1, S)$

  $(q_{4_y}, 1, q_{4_y}, 0, L)$

  $(q_{4_y}, B, q_5, 1, S)$

  $(q_{4_y}, *, q_{4_y}, *, L)$

- **Stage D** move right to x, y or B

  (q_5, x, q_6, x, L)

  (q_5, y, q_6, y, L)

  (q_5, B, q_6, B, L)

  $(q_5, *, q_5, *, R)$

- **Stage E** replace 0 by x, 1 by y

  $(q_6, 0, q_0, x, R)$

  $(q_6, 1, q_0, y, R)$

Turing Machine Examples

Binary Addition Example (7)

- **Stage F** replace x by 0, y by 1

  $(q_7, x, q_7, 0, L)$

  $(q_7, y, q_7, 1, L)$

  (q_7, B, q_h, B, R)

  $(q_7, *, q_7, *, L)$

- **Exercise** Evaluate $11 + 10 \mapsto 101$

Turing Machine Examples

Binary Addition Example (7a)

- **Exercise** Evaluate $11 + 10 \mapsto 101$
- **Stage A** find the rightmost symbol

$BBq_0 11B10B$ Note space symbols B at start and end

$\vdash BB1q_0 1B10B$

$\vdash BB11q_0 B10B$

$\vdash BB11Bq_1 10B$

$\vdash BB11B1q_1 0B$

$\vdash BB11B10q_1 B$

$\vdash BB11B1q_2 0B$

$\vdash BB11Bq_{3_x} 1BB$

- **Stage Bx** move left to blank

$\vdash B11q_{3_x} B1BB$

- **Stage Cx** move left to 0, 1, or blank

$\vdash BB1q_{4_x} 1B1BB$

$\vdash BB1Yq_0 B1BB$

Turing Machine Examples
Binary Addition Example (7b)

- **Exercise** Evaluate $11 + 10 \mapsto 101$ (contd)
- **Stage A** find the rightmost symbol

$\vdash BB1BYBq_1 1BB$

$\vdash BB1YB1q_1 BB$

$\vdash BB1YBq_2 1BB$

$\vdash BB1Yq_{3_y} BBBB$

- **Stage Cy** move left to 0, 1, or blank

$\vdash BB1q_{4_y} YBBBB$

$\vdash BBq_{4_y} 1YBBBB$

$\vdash Bq_{4_y} B0YBBBB$

$\vdash Bq_5 10YBBBB$

- **Stage D** move right to x, y or B

$\vdash Bq_5 0YBBBB$

$\vdash B0q_5 YBBBB$

$\vdash Bq_6 0YBBBB$

- **Stage E** replace 0 by x, 1 by y

$\vdash B1Xq_0 YBBBB$

Turing Machine Examples

Binary Addition Example (7c)

- **Exercise** Evaluate $11 + 10 \mapsto 101$ (contd)
- **Stage A** find the rightmost symbol

$\vdash \; B1XYq_0BBBB$

$\vdash \; B1XYBq_1BBB$

$\vdash \; B1XYq_2BBBB$

$\vdash \; B1Xq_7YBBBB$

- **Stage F** replace x by 0, y by 1

$\vdash \; B1q_7X1BBBB$

$\vdash \; Bq_7101BBBB$

$\vdash \; Bq_7B101BBBB$

$\vdash \; Bq_h101BBBB$

- This is mimicking what you learnt to do on paper as a child! Real *step-by-step* instructions
- See Morphett's Turing machine simulator for more examples (takes too long by hand!)

Computability

Universal Turing Machine

- Universal Turing Machine, U, is a Turing Machine that can simulate any arbitrary Turing machine, M
- Achieves this by encoding the transition function of M in some standard way
- The input to U is the encoding for M followed by the data for M
- See Turing machine examples

Computability

Decidability

- **Decidable** — there is a TM that will halt with yes/no for a decision problem — that is, given a string w over the alphabet of P the TM with halt and return yes.no the string is in the language P (same as *recursive* in Recursion theory — old use of the word)
- **Semi-decidable** — there is a TM will halt with yes if some string is in P but may loop forever on some inputs (same as *recursively enumerable*) — *Halting Problem*
- **Highly-undecidable** — no outcome for any input — *Totality, Equivalence Problems*

Computability

Undecidable Problems

- **Halting problem** — the problem of deciding, given a program and an input, whether the program will eventually halt with that input, or will run forever — term first used by Martin Davis 1952
- **Entscheidungsproblem** — the problem of deciding whether a given statement is provable from the axioms using the rules of logic — shown to be undecidable by Turing (1936) by reduction from the *Halting problem* to it
- **Type inference and type checking** in the second-order lambda calculus (important for functional programmers, Haskell, GHC implementation)
- Undecidable problem — see link to list

Computability

Halting Problem — Sketch Proof (1)

- Halting problem — is there a program that can determine if any arbitrary program will halt or continue forever ?
- Assume we have such a program (Turing Machine) `h(f,x)` that takes a program `f` and input `x` and determines if it halts or not

```
h(f,x)
  = if f(x) runs forever
      return True
    else
      return False
```

- We shall prove this cannot exist by contradiction

Computability

Halting Problem — Sketch Proof (2)

- Now invent two further programs:
- `q(f)` that takes a program `f` and runs `h` with the input to `f` being a copy of `f`
- `r(f)` that runs `q(f)` and halts if `q(f)` returns `True`, otherwise it loops

```
q(f)
  = h(f,f)

r(f)
  = if q(f)
      return
    else
      while True: continue
```

- What happens if we run `r(r)` ?
- If it loops, `q(r)` returns `True` and it does not loop — contradiction.

Computability

Why undecidable problems must exist

- A *problem* is really membership of a string in some language
- The number of different languages over any alphabet of more than one symbol is uncountable
- Programs are finite strings over a finite alphabet (ASCII or Unicode) and hence countable.
- There must be an infinity (big) of problems more than programs.
- **Computational problem** — defined by a function
- **Computational problem is computable** if there is a Turing machine that will calculate the function.

Computability

Computability and Terminology (1)

- The idea of an *algorithm* dates back 3000 years to Euclid, Babylonians...
- In the 1930s the idea was made more formal: which *functions* are *computable*?
- A *function* is a set of pairs $f = \{(x, f(x)) : x \in X \land f(x) \in Y\}$ with the *function property*
- *Function property:* $(a, b) \in f \land (a, c) \in f \Rightarrow b == c$
- *Function property:* Same input implies same output
- Note that maths notation is deeply inconsistent here — see Function and History of the function concept
- *What do we mean by computing a function — an algorithm ?*

Functions

Relation and Rule

- The idea of function as a set of pairs (Binary relation) with the function property (each element of the domain has at most one element in the co-domain) is fairly recent — see History of the function concept
- School maths presents us with function as rule to get from the input to the output
- Example: the square function: $\texttt{square}\ x = x \times x$
- But lots of rules (or algorithms) can implement the same function
- `square1 x = x^2`
- $\texttt{square2}\ x\ = \overbrace{x + \cdots + x}^{x\ \texttt{times}}$ if x is integer

Computability

Computability and Terminology (2)

- In the 1930s three definitions:
- λ-Calculus, simple semantics for computation — Alonzo Church
- General recursive functions — Kurt Gödel
- Universal (Turing) machine — Alan Turing
- Terminology:
  - Recursive, recursively enumerable — Church, Kleene
  - Computable, computably enumerable — Gödel, Turing
  - Decidable, semi-decidable, highly undecidable
  - In the 1930s, *computers* were *human*
  - Unfortunate choice of terminology
- Turing and Church showed that the above three were equivalent
- Church-Turing thesis — function is intuitively computable if and only if Turing machine computable

Computability

Reducing one problem to another

- To reduce problem P_1 to P_2, invent a construction that converts instances of P_1 to P_2 that have the same answer. That is:
  - any string in the language P_1 is converted to some string in the language P_2
  - any string over the alphabet of P_1 that is not in the language of P_1 is converted to a string that is not in the language P_2
- With this construction we can solve P_1
  - Given an instance of P_1, that is, given a string w that may be in the language P_1, apply the construction algorithm to produce a string x
  - Test whether x is in P_2 and give the same answer for w in P_1

Computability

Problem Reduction

- **Problem Reduction — Ordinary Example**
- Want to phone Alice but don't have her number
- You know that Bill has her number
- So *reduce* the problem of finding Alice's number to the problem of getting hold of Bill

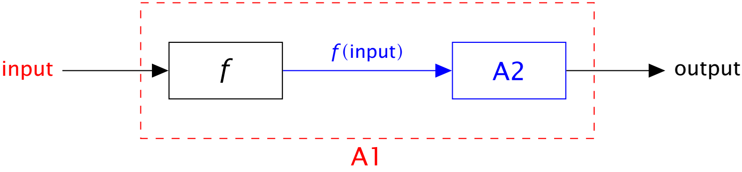
Computability

Direction of Reduction

- The direction of reduction is important
- If we can reduce P_1 to P_2 then (in some sense) P_2 is at least as hard as P_1 (since a solution to P_2 will give us a solution to P_1)
- So, if P_2 is decidable then P_1 is decidable
- To show a problem is undecidable we have to reduce from an known undecidable problem to it
- $\forall x(\mathrm{dp}_{P_1}(x) = \mathrm{dp}_{P_2}(\mathrm{reduce}(x)))$
- Since, if P_1 is undecidable then P_2 is undecidable

Reductions & Non-Computable

Reductions

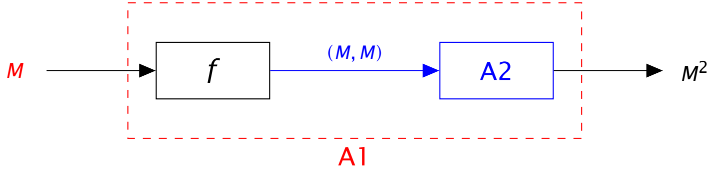


- A *reduction* of problem P_1 to problem P_2
  - transforms inputs to P_1 into inputs to P_2
  - runs algorithm $A2$ (which solves P_2) and
  - interprets the outputs from $A2$ as answers to P_1
- More formally: A problem P_1 is *reducible* to a problem P_2 if there is a function f that takes any input x to P_1 and transforms it to an input $f(x)$ of P_2

  such that the solution of P_2 on $f(x)$ is the solution of P_1 on x

Reductions & Non-Computible

Example: Squaring a Matrix



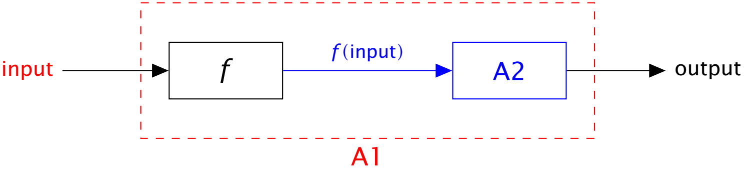
- Given an algorithm ($A2$) for matrix multiplication (P_2)
  - Input: pair of matrices, (M_1, M_2)
  - Output: matrix result of multiplying M_1 and M_2
- P_1 is the problem of squaring a matrix
  - Input: matrix M
  - Output: matrix M^2
- Algorithm $A1$ has

  $f(M) = (M, M)$

  uses $A2$ to calculate $M \times M = M^2$

Reductions & Non-Computable

Non-Computable Problems



- If P_2 is computable ($A2$ exists) then P_1 is computable (f being simple or polynomial)
- Equivalently If P_1 is non-computable then P_2 is non-computable
- **Exercise:** show $B \rightarrow A \equiv \neg A \rightarrow \neg B$

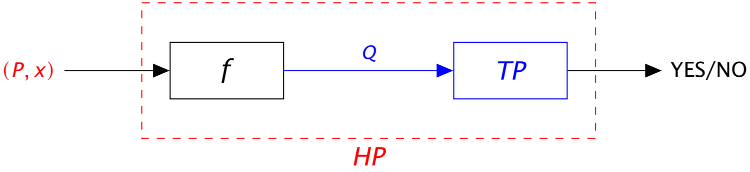
Reductions & Non-Computable

Contrapositive

- Proof by Contrapositive
- $B \to A \equiv \neg B \vee A$ by truth table or equivalences
  $\equiv \neg(\neg A) \vee \neg B$ commutativity and negation laws
  $\equiv \neg A \to \neg B$ equivalences
- Common error: switching the order round

Reductions & Non-Computable

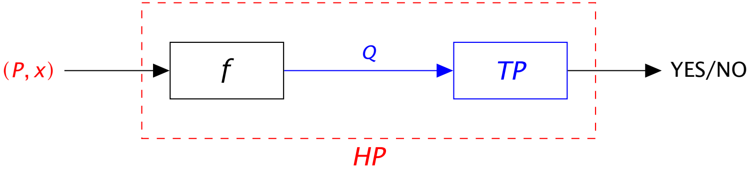
Totality Problem



- **Totality Problem**
  - Input: program Q
  - Output: YES if Q terminates for all inputs else NO
- Assume we have algorithm TP to solve the Totality Problem
- Now reduce the Halting Problem to the Totality Problem

Reductions & Non-Computable

Totality Problem



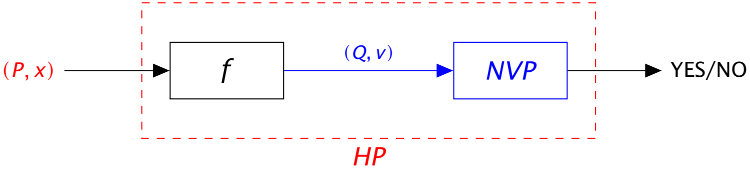
- Define f to transform inputs to HP to TP pseudo-Python

```
def f(P,x) :
  def Q(y):
    # ignore y
    P(x)
  return Q
```

- Run TP on Q
  - If TP returns YES then P halts on x
  - If TP returns NO then P does not halt on x
- We have *solved* the Halting Problem — contradiction

Reductions & Non-Computable

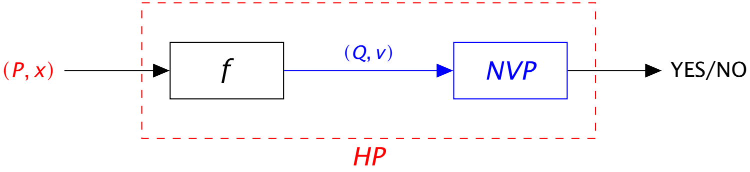
Negative Value Problem



- **Negative Value Problem**
  - Input: program Q which has no input and variable v used in Q
  - Output: YES if v ever gets assigned a negative value else NO
- Assume we have algorithm NVP to solve the Negative Value Problem
- Now reduce the Halting Problem to the Negative Value Problem

Reductions & Non-Computable

Negative Value Problem



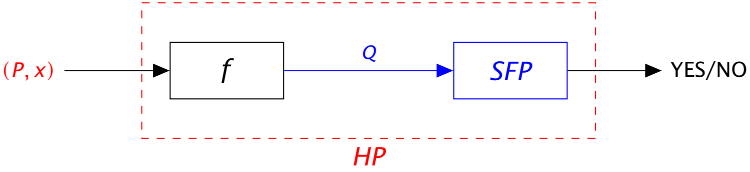
- Define f to transform inputs to HP to NVP pseudo-Python

```
def f(P,x) :
  def Q(y):
    # ignore y
    P(x)
    v = -1
  return (Q,var(v))
```

- Run NVP on $(Q, var(v))$ `var(v)` gets the variable name
  - If NVP returns YES then P halts on x
  - If NVP returns NO then P does not halt on x
- We have *solved* the Halting Problem — contradiction

Reductions & Non-Computable

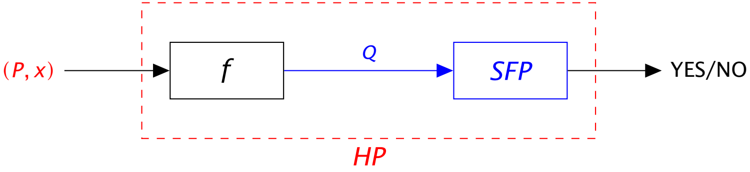
Squaring Function Problem



- **Squaring Function Problem**
  - Input: program Q which takes an integer, y
  - Output: YES if Q always returns the square of y else NO
- Assume we have algorithm *SFP* to solve the Squaring Function Problem
- Now reduce the Halting Problem to the Squaring Function Problem

Reductions & Non-Computable

Squaring Function Problem



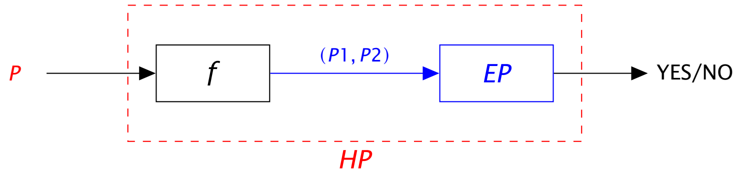
- Define f to transform inputs to HP to SFP pseudo-Python

```
def f(P,x) :
  def Q(y):
    P(x)
    return y * y
  return Q
```

- Run *SFP* on *Q*
  - If *SFP* returns YES then *P* halts on *x*
  - If *SFP* returns NO then *P* does not halt on *x*
- We have *solved* the Halting Problem — contradiction

Reductions & Non-Computable

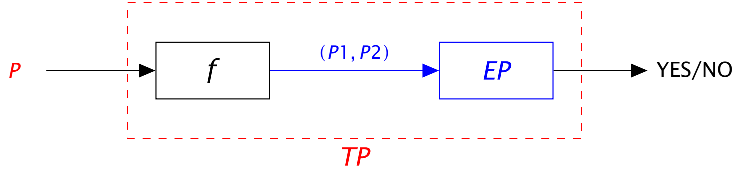
Equivalence Problem



- **Equivalence Problem**
  - Input: two programs $P1$ and $P2$
  - Output: YES if $P1$ and $P2$ solve the ame problem (same output for same input) else NO
- Assume we have algorithm EP to solve the Equivalence Problem
- Now reduce the Totality Problem to the Equivalence Problem

Reductions & Non-Computable

Equivalence Problem



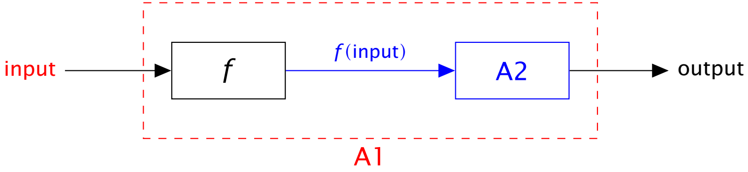
- Define f to transform inputs to TP to EP pseudo-Python

```
def f(P) :
  def P1(x):
    P(x)
    return "Same␣string"
  def P2(x)
    return "Same␣string"
  return (P1,P2)
```

- Run EP on $(P1, P2)$
  - If EP returns YES then P halts on all inputs
  - If EP returns NO then P does not halt on all inputs
- We have *solved* the Totality Problem — contradiction

Reductions & Non-Computable

Rice's Theorem



- Rice's Theorem all non-trivial, semantic properties of programs are undecidable. H G Rice 1951 PhD Thesis
- Equivalently: For any non-trivial property of partial functions, no general and effective method can decide whether an algorithm computes a partial function with that property.
- A property of partial functions is called trivial if it holds for all partial computable functions or for none.

Reductions & Non-Computable

Rice's Theorem

- Rice's Theorem and computability theory
- Let S be a set of languages that is nontrivial, meaning
  - there exists a Turing machine that recognizes a language in S
  - there exists a Turing machine that recognizes a language not in S
- Then, it is undecidable to determine whether the language recognized by an arbitrary Turing machine lies in S.
- This has implications for compilers and virus checkers
- Note that Rice's theorem does not say anything about those properties of machines or programs that are not also properties of functions and languages.
- For example, whether a machine runs for more than 100 steps on some input is a decidable property, even though it is non-trivial.

Lambda Calculus

Motivation

- Lambda Calculus is a formal system in mathematical logic for expressing computation based on function abstraction and application using variable binding and substitution
- Lambda calculus is Turing complete — it can simulate any Turing machine
- Introduced by Alonzo Church in 1930s
- Basis of functional programming languages — Lisp, Scheme, ISWIM, ML, SASL, KRC, Miranda, Haskell, Scala, F#...
- **Note** this is not part of M269 but may help understand ideas of computability

Functions

Binding and Substitution

- School maths introduces functions as $f(x) = 3x^2 + 4x + 5$
- Substitution: $f(2) = 3 \times 2^2 + 4 \times 2 + 5 = 25$
- Generalise: $f(x) = ax^2 + bx + c$
- What is wrong with the following:
- $f(a) = a \times a^2 + b \times a + c$
- The ideas of free and bound variables and substitution
- Evaluating an expression — how many ways can you evaluate $(3 + 7)^2$
- Answer: 3 ways (Bird, 1998, Ex 1.2.2, page 6)
- Ways of evaluating $((3 + 7)^2)^2$
- Answer: 547 ways (Bird and Wadler, 1988, Ex 1.2.1, page 6)

Lambda Calculus

Optional Topic

- M269 Unit 6/7 Reader *Logic and the Limits of Computation* alludes to other formalisations with equal power to a Turing Machine (pages 81 and 87)
- The *Reader* mentions Alonzo Church and his 1930s formalism (page 87, but does not give any detail)
- The notes in this section are optional and for comparison with the Turing Machine material
- Turing machine: explicit memory, state and implicit loop and case/if statement
- Lambda Calculus: function definition and application, explicit rules for evaluation (and transformation) of expressions, explicit rules for substitution (for function application)
- Lambda calculus reduction workbench

Lambda Calculus

Lambda Terms

- A **variable**, x, is a lambda term
- If M is a lambda term and x is a variable, then $(\lambda x.M)$ is a lambda term — a **lambda abstraction** or function definition
- If M and N are lambda terms, the $(M\ N)$ is lambda term — an **application**
- Nothing else is a lambda term

Lambda Calculus

Lambda Terms — Notational Conveniences

- Outermost parentheses are omitted $(M\ N) \equiv M\ N$
- Application is left associative $((M\ N)\ P) \equiv M\ N\ P$
- The body of an abstraction extends as far right as possible, subject to scope limited by parentheses
- $\lambda x.M\ N \equiv \lambda x.(M\ N)$ and not $(\lambda x.M)\ N$
- $\lambda x.\lambda y.\lambda z.M \equiv \lambda x\,y\,z.M$

Lambda Calculus

Lambda Calculus Semantics

- What do we mean by *evaluating an expression* ?
- To evaluate $(\lambda x.M)N$
- Evaluate M with x replaced by N
- This rule is called β-reduction
- $(\lambda x.M)N \underset{\beta}{\rightarrow} M[x := N]$
- $M[x := N]$ is M with occurrences of x replaced by N
- This operation is called *substitution* — see rules below

Lambda Calculus

β-Reduction Examples

- $(\lambda x.x)z \to z$
- $(\lambda x.y)z \to y$
- $(\lambda x.x\,y)z \to z\,y$

  a function that applies its argument to y
- $(\lambda x.x\,y)(\lambda z.z) \to (\lambda z.z)y \to y$
- $(\lambda x.\lambda y.x\,y)z \to \lambda y.z\,y$

  A *curried* function of two arguments — applies first argument to second
- *currying* replaces $f(x, y)$ with $(f\,x)y$ — nice notational convenience — gives *partial application* for free

Lambda Calculus

Substitution

- To define *substitution* use recursion on the structure of terms
- $x[x := N] \equiv N$
- $y[x := N] \equiv y$
- $(P\,Q)[x := N] \equiv (P[x := N])\,(Q[x := N])$
- $(\lambda x.M)[x := N] = \lambda x.M$

  In $(\lambda x.M)$, the x is a formal parameter and thus a local variable, different to any other
- $(\lambda y.M)[x := N] =$ what?
- Look back at the school maths example above — a subtle point

Lambda Calculus

Substitution (2)

- Renaming *bound variables consistently is allowed*
- $\lambda x.x \equiv \lambda y.y \equiv \lambda z.z$
- $\lambda y.\lambda x.y \equiv \lambda z.\lambda x.z$
- This is called α*-conversion*
- $(\lambda x.\lambda y.x\,y)\,y \rightarrow (\lambda x.\lambda z.x\,z)\,y \rightarrow \lambda z.y\,z$

Lambda Calculus

Substitution (3)

- **Bound and Free Variables**
- $BV(x) = \emptyset$
- $BV(\lambda x.M) = BV(M) \cup \{x\}$
- $BV(M\,N) = BV(M) \cup BV(N)$
- $FV(x) = \{x\}$
- $FV(\lambda x.M) = FV(M) - \{x\}$
- $FV(M\,N) = FV(M) \cup FV(N)$
- The above is a formalisation of school maths
- A Lambda term with no free variables is said to be *closed* — such terms are also called **combinators** — see Combinator and Combinatory logic (Hankin, 2004, page 10)
- α-**conversion**
- $\lambda x.M \underset{\alpha}{\rightarrow} \lambda y.M[x := y]$ if $y \notin FV(M)$

Lambda Calculus
Substitution (4)

- β-reduction final rule
- $(\lambda y.M)[x := N] = \lambda y.M$ if $x \notin FV(M)$
- $(\lambda y.M)[x := N] = \lambda y.M[x := N]$
  if $x \in FV(M)$ and $y \notin FV(N)$
- $(\lambda y.M)[x := N] = \lambda z.M[y := z][x := N]$
  if $x \in FV(M)$ and $y \in FV(N)$
  z is chosen to be first variable $z \notin FV(N\,M)$
- This is why you cannot go $f(a)$ when given
- $f(x) = ax^2 + bx + c$
- School maths — but made formal

Lambda Calculus

Rules Summary — Conversion

- α-**conversion** renaming bound variables
- $\lambda x.M \underset{\alpha}{\rightarrow} \lambda y.M[x := y]$ if $y \notin FV(M)$
- β-**conversion** function application
- $(\lambda x.M)N \underset{\beta}{\rightarrow} M[x := N]$
- η-**conversion** extensionality
- $\lambda x.F\,x \underset{\eta}{\rightarrow} F$ if $x \notin FV(F)$

Lambda Calculus

Rules Summary — Substitution

1. $x[x := N] \equiv N$
2. $y[x := N] \equiv y$
3. $(P\,Q)[x := N] \equiv (P[x := N])\,(Q[x := N])$
4. $(\lambda x.M)[x := N] = \lambda x.M$
5. $(\lambda y.M)[x := N] = \lambda y.M$ if $x \notin FV(M)$
6. $(\lambda y.M)[x := N] = \lambda y.M[x := N]$

   if $x \in FV(M)$ and $y \notin FV(N)$
7. $(\lambda y.M)[x := N] = \lambda z.M[y := z][x := N]$

   if $x \in FV(M)$ and $y \in FV(N)$

   z is chosen to be first variable $z \notin FV(N\,M)$

Lambda Calculus

Lambda Calculus Encodings

- So what does this formalism get us ?
- The Lambda Calculus is Turing complete
- We can encode any computation (if we are clever enough)
- Booleans and propositional logic
- Pairs
- Natural numbers and arithmetic
- Looping and recursion

Lambda Calculus Encodings

Booleans and Propositional Logic

- True $= \lambda x.\lambda y.x$
- False $= \lambda x.\lambda y.y$
- IF a THEN b ELSE $c \equiv a\,b\,c$
- IF True THEN b ELSE $c \rightarrow (\lambda x.\lambda y.x)\ b\ c$
- $\rightarrow (\lambda y.b)\ c \rightarrow b$
- IF False THEN b ELSE $c \rightarrow (\lambda x.\lambda y.y)\ b\ c$
- $\rightarrow (\lambda y.y)\ c \rightarrow c$

Lambda Calculus Encodings

Booleans and Propositional Logic (2)

- Not $= \lambda x.((x\,\text{False})\text{True})$
- Not x = IF x THEN False ELSE True
- Exercise: evaluate Not True
- And $= \lambda x.\lambda y.((x\ y)\ \text{False})$
- And x y = IF x THEN y ELSE False
- Exercise: evaluate And True False
- Or $= \lambda x.\lambda y.((x\ \text{True}\ )\ y)$
- Or x y = IF x THEN True ELSE y
- Exercise: evaluate Or False True

Lambda Calculus Encodings

Booleans and Propositional Logic (2) — Exercises

- Exercise: evaluate Not True
- $\rightarrow (\lambda x.((x \ \text{False}) \ \text{True})) \ \text{True}$
- $\rightarrow (\text{True} \ \ \text{False}) \ \text{True}$
- Could go straight to False from here, but we shall fill in the detail
- $\rightarrow ((\lambda x.\lambda y.x) \ (\lambda x.\lambda y.y)) \ (\lambda x.\lambda y.x)$
- $\rightarrow (\lambda y.(\lambda x.\lambda y.y)) \ (\lambda x.\lambda y.x)$
- $\rightarrow (\lambda x.\lambda y.y) \equiv$ False
- Exercise: evaluate And True False
- $\rightarrow$(IF x THEN y ELSE False) True False
- $\rightarrow$(IF True THEN False ELSE False) $\rightarrow$False
- Exercise: evaluate Or False True
- $\rightarrow$(IF x THEN True ELSE y) False True
- $\rightarrow$(IF False THEN True ELSE True) $\rightarrow$True

Lambda Calculus Encodings

Natural Numbers — Church Numerals

- Encoding of natural numbers
- $0 = \lambda f.\lambda y.y$
- $1 = \lambda f.\lambda y.f\ y$
- $2 = \lambda f.\lambda y.f\ (f\ y)$
- $3 = \lambda f.\lambda y.f\ (f\ (f\ y))$
- Successor Succ $= \lambda z.\lambda f.\lambda y.f(z\ f\ y)$
- Succ $0 = (\lambda z.\lambda f.\lambda y.f(z\ f\ y))(\lambda f.\lambda y.y)$
- $\rightarrow \lambda f.\lambda y.f\ ((\lambda f.\lambda y.y)\ f\ y)$
- $\rightarrow \lambda f.\lambda y.f\ ((\lambda y.y)\ y)$
- $\rightarrow \lambda f.\lambda y.f\ y = 1$

Lambda Calculus Encodings

Natural Numbers — Operations

- isZero $= \lambda z.z(\lambda y.\text{ False })\text{ True}$
- Exercise: evaluate isZero 0
- If M and N are numerals (as λ expressions)
- Add $M\,N = \lambda x.\lambda y.(M\,x)\,((N\,x)\,y)$
- Mult $M\,N = \lambda x.(M\,(N\,x))$
- Exercise: show 1 + 1 = 2

Lambda Calculus Encodings

Pairs

- Encoding of a pair a, b
- $(a, b) = \lambda x.$ IF x THEN a ELSE b
- FST $= \lambda f . f$ True
- SND $= \lambda f . f$ False
- Exercise: evaluate FST (a, b)
- Exercise: evaluate SND (a, b)

Lambda Calculus Encodings

The Fixpoint Combinator

- $Y = \lambda f.(\lambda x.f\,(x\,x))\,(\lambda x.f\,(x\,x))$
- $Y\,F = \lambda f.(\lambda x.f\,(x\,x))\,(\lambda x.f\,(x\,x))\,F$
- $\rightarrow (\lambda x.F\,(x\,x))(\lambda x.F\,(x\,x))$
- $F((\lambda x.F\,(x\,x))\,(\lambda x.F\,(x\,x))) = F\,(Y\,F)$
- $(Y\,F)$ is a fixed point of F
- We can use Y to achieve recursion for F

Lambda Calculus Encodings

The Fixpoint Combinator — Recursion

- **Recursion implementation — Factorial**
- Fact $= \lambda f.\lambda n.$ IF $n = 0$ THEN 1 ELSE $n * (f\,(n-1))$
- $(Y$ Fact$)1 = ($Fact $(Y$ Fact$))1$
- $\rightarrow$ IF $1 = 0$ THEN 1 ELSE $1 * ((Y$ Fact$)\,0)$
- $\rightarrow 1 * ((Y$ Fact$)\,0)$
- $\rightarrow 1 * ($Fact $(Y$ Fact$)\,0)$
- $\rightarrow 1 *$ IF $0 = 0$ THEN 1 ELSE $0 * ((Y$ Fact$)\,(0-1))$
- $\rightarrow 1 * 1 \rightarrow 1$
- Factorial n = (Y Fact) n
- Recursion implemented with a non-recursive function Y

Computability

Turing Machines, Lambda Calculus and Programming Languages

- Anything computable can be represented as TM or Lambda Calculus
- But programs would be slow, large and hard to read
- In practice use the ideas to create more expressive languages which include built-in primitives
- Also leads to ideas on data types
- Polymorphic data types
- Algebraic data types
- Also leads on to ideas on higher order functions — functions that take functions as arguments or returns functions as results.

Complexity

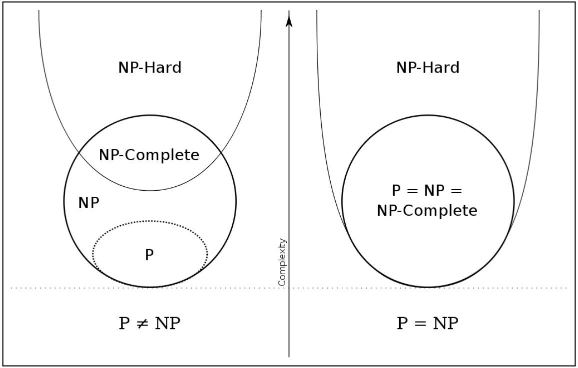
P and NP

- P, the set of all decision problems that can be solved in polynomial time on a deterministic Turing machine
- NP, the set of all decision problems whose solutions can be verified (certificate) in polynomial time
- Equivalently, NP, the set of all decision problems that can be solved in polynomial time on a non-deterministic Turing machine
- A *decision problem*, *dp* is *NP-complete* if
  1. *dp* is in NP and
  2. Every problem in NP is reducible to *dp* in polynomial time
- NP-hard — a problem satisfying the second condition, whether or not it satisfies the first condition. Class of problems which are at least as hard as the hardest problems in NP. NP-hard problems do not have to be in NP and may not be decision problems

Complexity

P and NP — Diagram

Euler diagram for P, NP, NP-complete and NP-hard set of problems



Source: Wikipedia NP-complete entry

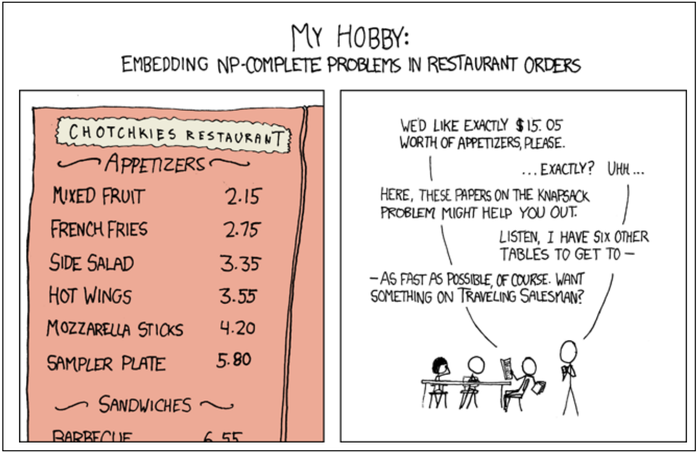
Complexity

NP-complete problems

- Boolean satisfiability (SAT) Cook-Levin theorem
- Conjunctive Normal Form 3SAT
- Hamiltonian path problem
- Travelling salesman problem
- NP-complete — see list of problems

Complexity

Knapsack Problem



Source & Explanation: XKCD 287

NP-Completeness and Boolean Satisfiability

Points on Notes

- The *Boolean satisfiability problem (SAT)* was the first decision problem shown to be *NP-Complete*
- This section gives a sketch of an explanation
- **Health Warning** different texts have different notations and there will be some inconsistency in these notes
- **Health warning** these notes use some formal notation *to make the ideas more precise* — computation requires precise notation and is about manipulating strings according to precise rules.

NP-Completeness and Boolean Satisfiability

Alphabets, Strings and Languages

- Notation:
- Σ is a set of symbols — the alphabet
- Σ^k is the set of all string of length k, which each symbol from Σ
- Example: if $\Sigma = \{0, 1\}$
  - $\Sigma^1 = \{0, 1\}$
  - $\Sigma^2 = \{00, 01, 10, 11\}$
- $\Sigma^0 = \{\epsilon\}$ where ϵ is the empty string
- Σ^* is the set of all possible strings over Σ
- $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$
- A *Language*, L, over Σ is a subset of Σ^*
- $L \subseteq \Sigma^*$

NP-Completeness and Boolean Satisfiability

Language Accepted by a Turing Machine

- Language accepted by Turing Machine, M denoted by $L(M)$
- $L(M)$ is the set of strings $w \in \Sigma^*$ accepted by M
- For *Final States* $F = \{Y, N\}$, a string $w \in \Sigma^*$ is accepted by $M \Leftrightarrow$ (if and only if) M starting in q_0 with w on the tape halts in state Y
- Calculating a function (function problem) can be turned into a decision problem by asking whether $f(x) = y$

NP-Completeness and Boolean Satisfiability

The NP-Complete Class

- If we do not know if P $\neq$ NP, what can we say ?
- A language L is *NP-Complete* if:
  - $L \in$ NP and
  - for all other $L' \in$ NP there is a *polynomial time transformation* (Karp reducible, reduction) from L' to L
- Problem P_1 *polynomially reduces* (Karp reduces, transforms) to P_2, written $P_1 \propto P_2$ or $P_1 \leq_p P_2$, iff $\exists f : \mathrm{dp}_{P_1} \rightarrow \mathrm{dp}_{P_2}$ such that
  - $\forall I \in \mathrm{dp}_{P_1} [I \in Y_{P_1} \Leftrightarrow f(I) \in Y_{P_2}]$
  - f can be computed in polynomial time

NP-Completeness and Boolean Satisfiability

The NP-Complete Class (2)

- More formally, $L_1 \subseteq \Sigma_1^*$ polynomially transforms to $L_2 \subseteq \Sigma_2^*$, written $L_1 \propto L_2$ or $L_1 \leq_p L_2$, iff $\exists f : \Sigma_1^* \to \Sigma_2^*$ such that
  - $\forall x \in \Sigma_1^* [x \in L_1 \Leftrightarrow f(x) \in L_2]$
  - There is a polynomial time TM that computes f
- *Transitivity* If $L_1 \propto L_2$ and $L_2 \propto L_3$ then $L_1 \propto L_3$
- If L is NP-Hard and $L \in$ P then P = NP
- If L is NP-Complete, then $L \in$ P if and only if P = NP
- If L_0 is NP-Complete and $L \in$ NP and $L_0 \propto L$ then L is NP-Complete
- Hence if we find one NP-Complete problem, it may become easier to find more
- In 1971/1973 Cook-Levin showed that the Boolean satisfiability problem (SAT) is NP-Complete

NP-Completeness and Boolean Satisfiability

The Boolean Satisfiability Problem

- A propositional logic formula or Boolean expression is built from variables, operators: AND (conjunction, $\wedge$), OR (disjunction, $\vee$), NOT (negation, $\neg$)
- A formula is said to be *satisfiable* if it can be made True by some assignment to its variables.
- *The Boolean Satisfiability Problem* is, given a formula, check if it is satisfiable.
  - *Instance:* a finite set U of Boolean variables and a finite set C of clauses over U
  - *Question:* Is there a satisfying truth assignment for C ?
- A *clause* is is a disjunction of variables or negations of variables
- *Conjunctive normal form (CNF)* is a conjunction of clauses
- Any Boolean expression can be transformed to CNF

NP-Completeness and Boolean Satisfiability

The Boolean Satisfiability Problem (2)

- Given a set of Boolean variable $U = \{u_1, u_2, \ldots, u_n\}$
- A literal from U is either any u_i or the negation of some u_i (written $\overline{u_i}$)
- A clause is denoted as a subset of literals from U — $\{u_2, \overline{u_4}, u_5\}$
- A clause is satisfied by an assignment to the variables if at least one of the literals evaluates to True (just like disjunction of the literals)
- Let C be a set of clauses over U — C is satisfiable iff there is some assignment of truth values to the variables so that every clause is satisfied (just like CNF)
- $C = \{\{u_1, u_2, u_3\}, \{\overline{u_2}, \overline{u_3}\}, \{u_2, \overline{u_3}\}\}$ is satisfiable
- $C = \{\{u_1, u_2\}, \{u_1, \overline{u_2}\}, \{\overline{u_1}\}\}$ is not satisfiable

NP-Completeness and Boolean Satisfiability

The Boolean Satisfiability Problem (3)

- Proof that SAT is NP-Complete looks at the structure of NDTMs and shows you can transform any NDTM to SAT in polynomial time (in fact logarithmic space suffices)
- SAT is in NP since you can check a solution in polynomial time
- To show that $\forall L \in \mathrm{NP} : L \propto \mathrm{SAT}$ invent a polynomial time algorithm for each polynomial time NDTM, M, which takes as input a string x and produces a Boolean formula E_x which is satisfiable iff M accepts x
- See Cook-Levin theorem

NP-Completeness and Boolean Satisfiability

Coping with NP-Completeness

- What does it mean if a problem is NP-Complete ?
  - There is a P time verification algorithm.
  - There is a P time algorithm to solve it iff P = NP (?)
  - No one has yet found a P time algorithm to solve any NP-Complete problem
  - So what do we do ?
- Improved exhaustive search — Dynamic Programming; Branch and Bound
- Heuristic methods — *acceptable* solutions in *acceptable* time — compromise on optimality
- Average time analysis — look for an algorithm with good average time — compromise on generality (see Big-O Algorithm Complexity Cheatsheet)
- Probabilistic or Randomized algorithms — compromise on correctness

Future Work

Topics & Events

- Sunday, 16 May 2021 online tutorial exam revision
- Saturday, 22 May 2021 tutorial online, exam revision
- Please email me with any requests for particular topics
- Tuesday, 8 June 2021 exam

Web Sites

Computability

- **Logic**
  - WFF, WFF'N Proof online
- **Computability**
  - Computability
  - Computable function
  - Decidability (logic)
  - Turing Machines
  - Universal Turing Machine
  - Turing machine simulator
  - Lambda Calculus
  - Von Neumann Architecture
  - Turing Machine XKCD 205 Candy Button Paper
  - Turing Machine XKCD 505 A Bunch of Rocks
  - RIP John Conway Why can Conway's Game of Life be classified as a universal machine?
  - Phil Wadler Bright Club on Computability
  - Bridges: Theory of Computation: Halting Problem
  - Bridges: Theory of Computation: Other Non-computable Problems

Web Sites

Complexity

- **Complexity**
  - Complexity class
  - NP complexity
  - NP complete
  - Reduction (complexity)
  - P versus NP problem
  - Graph of NP-Complete Problems